PROBLEMS ON SYMPLECTIC REFLECTION ALGEBRAS

17. Procesi bundles and their deformations

Problem 17.1. *Let X be an algebraic variety, $\mathcal{F}_0$ be a coherent sheaf on X and $\mathcal{D}$ be a FCS (=flat, complete and separated) deformation of $\mathcal{O}_X$ over $\mathbb{C}[[\hbar]]$.*

(1) *Show that the category of finitely generated modules (i.e., sheaves) over $\mathcal{D}/(\hbar^n)$ has enough injective objects. How are the injectives for different n related?*
(2) *Show that if $\mathrm{Ext}^2(\mathcal{F}_0, \mathcal{F}_0) = 0$, then there exists a flat deformation of $\mathcal{F}_0$ to a right module $\mathcal{F}_n$ over $\mathcal{D}/(\hbar^{n+1})$. Moreover, show that these deformations may be chosen in a compatible way and so give rise to a FCS deformation $\mathcal{F}$ of $\mathcal{F}_0$ to a right module over $\mathcal{D}$.*
(3) *Finally, show that if $\mathrm{Ext}^1(\mathcal{F}_0, \mathcal{F}_0) = 0$, then all the deformations above are unique.*

Exercise 17.1. [1] *Let V_1, V_2 are $\mathbb{C}[[\mathfrak{z}^*, \hbar]]$-modules that are flat, complete and separated. Let $\iota : V_1 \to V_2$ be a $\mathbb{C}[[\mathfrak{z}^*, \hbar]]$-module homomorphism that is an isomorphism modulo $(\mathfrak{z}, \hbar)$. Show that ι is an isomorphism.*

Exercise 17.2. *Show that any fiber of a Procesi bundle is isomorphic to $\mathbb{C}\Gamma_n$ as a Γ_n-module.*

Problem 17.2. *Show that the dual of a Procesi bundle is again a Procesi bundle.*

Exercise 17.3. *Let A_0 be a $\mathbb{Z}_{\geqslant 0}$-graded vector space and A be its FCS deformation over $\mathbb{C}[[x_1, \ldots, x_n]]$. Equip A with a $\mathbb{C}^\times$-action such that $t.(x_i a) = t^2 x_i t.a$ and the projection $A \twoheadrightarrow A_0$ is $\mathbb{C}^\times$-equivariant (where the action of $\mathbb{C}^\times$ on the ith component of $\mathcal{A}_0$ is by $t \mapsto t^i$). Show that the $\mathbb{C}^\times$-finite part of A is a graded deformation of A_0 over $\mathbb{C}[x_1, \ldots, x_n]$.*

[1] Also appeared last time