PROBLEMS ON SYMPLECTIC REFLECTION ALGEBRAS

2. CBH algebras

Exercise 2.1. *Let $\varphi : \mathcal{A} \twoheadrightarrow \mathcal{A}'$ be an algebra epimorphism. Suppose $\mathcal{A}$ is filtered. Check that $\mathcal{A}'^{\leqslant n} := \varphi(\mathcal{A}^{\leqslant n})$ defines an algebra filtration on $\mathcal{A}'$.*

Problem 2.1. *Show that the monomials $x^i y^j, i, j \geqslant 0$, form a basis of the Weyl algebra $\mathrm{W}_2 = \mathbb{C}\langle x, y\rangle/(xy - yx - 1)$. For this, construct a representation of W_2 on $\mathbb{C}[x]$.*

Exercise 2.2. *Check that the product on the associated graded $\operatorname{gr} \mathcal{A}$ of a filtered algebra $\mathcal{A}$ is associative and has a unit.*

Exercise 2.3. *Let A be a graded algebra. Take a two-sided ideal $I \subset A$ and let $\operatorname{gr} I$ denote the span of the top degree parts of elements of I. Show that $\operatorname{gr} I$ is a two-sided ideal of A and identify $\operatorname{gr}(A/I)$ with $A/\operatorname{gr} I$.*

Problem 2.2. *Establish natural isomorphisms $R_h(\mathcal{A})/hR_h(\mathcal{A}) \cong \operatorname{gr} \mathcal{A}, R_h(\mathcal{A})/(h-\alpha)R_h(\mathcal{A}) \cong \mathcal{A}$, where $\alpha \in \mathbb{C} \setminus \{0\}$. Also check that $R_h(\mathcal{A})$ is flat over $\mathbb{C}[h]$.*

Problem 2.3. *Let A be a commutative associative algebra without zero divisors equipped with an action of a finite group Γ by automorphisms. We assume that the action is faithful meaning that only the unit acts trivially. Check that the map $a \mapsto a \otimes 1$ identifies A^Γ with the center of $A\#\Gamma$.*

Problem 2.4. *Show that if $\operatorname{gr}(\mathbb{C}\langle x, y\rangle\#\Gamma/(xy - yx - c)) = \mathbb{C}[x, y]\#\Gamma$, then c lies in the center of $\mathbb{C}\Gamma$ (that is equal to $(\mathbb{C}\Gamma)^\Gamma$, the invariants for the adjoint action).*

Exercise 2.4. *Deduce $\operatorname{gr} eH_c e = \mathbb{C}[x, y]^\Gamma$ from $\operatorname{gr} H_c = \mathbb{C}[x, y]\#\Gamma$ (i.e., show that taking the spherical subalgebra commutes with taking the associated graded).*

Problem 2.5. *Let Γ be the group $\mathbb{Z}/(r + 1)\mathbb{Z}$. We write $x, y \in H_c$ for the images of $x, y \in \mathbb{C}\langle x, y\rangle\#\Gamma$.*

1) Show that H_c is $\mathbb{Z}$-graded with Γ in degree 0, x in degree 1 and y in degree -1.

2) We can write c as $\sum_{\gamma\in\Gamma} c_\gamma \gamma$. Produce an element $h \in (H_c)^{\leqslant 2}$ that commutes with Γ and satisfies $[h, x] = c_1 x, [h, y] = -c_1 y$ (such an element is defined uniquely up to adding a constant provided $c_1 \neq 0$).

3) Set $x_1 := eh, x_2 := ex^{r+1}, x_3 := ey^{r+1}$. Check that there are polynomials P, Q in one variable of degree $r + 1$ such that $x_2 x_3 = P(x_1), x_3 x_2 = Q(x_1)$ in $eH_c e$. How are these polynomials related? Express their coefficients via the coefficients c_γ.

4) Use $\operatorname{gr} eH_c e = \mathbb{C}[x, y]^\Gamma$ to show that $eH_c e = \mathbb{C}\langle x_1, x_2, x_3\rangle/([x_1, x_2] = (r+1)c_1 x_2, [x_1, x_3] = -(r + 1)c_1 x_3, x_2 x_3 = P(x_1), x_3 x_2 = Q(x_1))$.

Exercise 2.5. *Prove that there are no non-constant invariant polynomials for the action of the one-dimensional torus $\mathbb{C}^\times$ on $\mathbb{C}^n$ given by $t.(x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$.*

Exercise 2.6. *Use the theorem (the only statement called this way in the lecture) to show that the closure of any orbit of a reductive group action on an affine variety contains a unique closed orbit.*

Problem 2.6. *Show that the algebra of invariants* $\mathbb{C}[X]^G$*, where* $X = \mathrm{Mat}_n(\mathbb{C})$ *and* $G = \mathrm{GL}_n(\mathbb{C})$ *acts on* X *by conjugations, is generated by the coefficients of the characteristic polynomial of a matrix and is isomorphic to the algebra of polynomials in* n *variables. A hint: consider the restriction to the subspace of diagonal matrices.*

Problem 2.7. *In the setting of the previous problem, check directly that every fiber indeed contains a single closed orbit and that this orbit consists of diagonalizable matrices.*