DIFFERENTIAL EQUATIONS ON HYPERPLANE COMPLEMENTS

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Contents

1. Introduction

The goal of the next two talks is to

(1) discuss a class of integrable connections associated to root systems
(2) describe their monodromy in terms of quantum groups

These connections come in two forms:

- Rational form leading to representations of braid groups (this week)
- Trigonometric form leading to representations of affine braid groups (next week)

The relevance of these connections is that

A: the quantum differential equations for Nakajima quiver varieties are of trigonometric type

B: the description of their monodromy in terms of quantum groups constitutes a step towards proving Roman Bezrukanikov's conjectures that the monodromy lifts to/comes from a braid group action on the derived category.

2. Flat connections on hyperplane complements

Reference of this section is section 2 of [14]. Rational connections are more generally associated to hyperplane arrangements which are not necessarily of root type. So let's consider this more general case.

Let $\mathcal{B}$ be a finite dimensional complex vector space, and $\mathcal{A} = \{\mathcal{H}_i\}_{i\in I}$ be a finite collection of linear hyperplanes.

Let $X = \mathcal{B} \setminus \mathcal{A}$ be the hyperplane complement.

Consider trivial vector bundles $\mathcal{V} = X \times \mathcal{F}$ on X, with fibers $\mathcal{F}$ a finite dimensional vector space. We consider the following meromorphic connection on the trivial vector bundle $\mathcal{V} = X \times \mathcal{F}$ over X:

$$\nabla = d - \sum_{i\in I} \frac{d\phi_i}{\phi_i} r_i, \tag{1}$$

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where $\phi_i \in \mathcal{B}^*$ are linear functions on $\mathcal{B}$, such that $\mathcal{H}_i = \mathrm{Ker}(\phi_i)$, and $r_i \in \mathrm{End}\,\mathcal{F}$ are called residues.

The following lemma gives a criterion for flatness of above connection, see [8].

Lemma 2.1 (Kohno)**.** *The above connection is flat if and only if for any subcollection of linear forms $\{\phi_j\}_{i\in J}$ which is maximal for the property that their span in $\mathcal{B}^*$ is two dimensional, one has*

$$[r_j, \sum_{j'\in J} r_{j'}] = 0,$$

for any $j \in J$.

Proof. Necessity of the criterion: since the form $\frac{d\phi_i}{\phi_i}$ are closed, the curvature Ω of the above connection is

$$\Omega = \frac{1}{2}\sum_{\phi_i,\phi_j} \frac{d\phi_i}{\phi_i} \wedge \frac{d\phi_j}{\phi_j}[r_i, r_j]$$

In general, let $H \subset X$ be a hyperplane defined by $s = 0$. We shall need to define the residue along H of a 2-form on X.

Lemma 2.2. *Let ϕ be a closed regular form on $X \setminus H$, with a polar singularity of order 1 along H. Then, there exist regular forms ψ, and θ of H, such that*

$$\phi = \frac{ds}{s} \wedge \psi + \theta$$

Note that $\psi|_H$ is a closed form only depends on ϕ, we call $\psi|_H$ the residue form of ϕ along the hyperplane H.

Take the residue of Ω along $\{\phi_i = 0\}$, we get a one-form, then take the residue of the one form $\mathrm{res}_{\{\phi_i=0\}}$ along $\{\phi_j = 0\}$, we get a function $[r_i, \sum_{j'\in J} r_{j'}]$, where J is spanned by ϕ_i, ϕ_j. Thus, $\Omega = 0$ implies the vanishing of $[r_i, \sum_{j'\in J} r_{j'}]$.

Sufficiency of the criterion:

Let us denote by Π the set of two dimensional subspaces of $\mathcal{B}^*$ spanned by subsets of $X = \{\phi_i\}_{i\in I}$. Clearly Π is in bijection with the set of all subcollections $J \subset I$, maximal with respect to property given in the statement of the lemma.

Define for any $\pi \in \Pi$,

$$\Omega^\pi = \sum_{x,y\in\pi\cap X} \frac{dx}{x} \wedge \frac{dy}{y}[r_x, r_y],$$

so we have $\Omega = \sum_{\pi\in\Pi} \Omega^\pi$.

Let $B = \langle x; y\rangle$ be a two dimensional space and let $z \in B$ be a non-zero vector. Then a simple calculation shows that

$$\frac{dx}{x} \wedge \frac{dy}{y} = \frac{dx}{x} \wedge \frac{dz}{z} + \frac{dz}{z} \wedge \frac{dy}{y}$$

For any $\pi \in \Pi$, fix a non-zero element z^π,

$$\begin{aligned}\Omega^\pi &= \sum_{x,y\in X\cap\pi} \frac{dx}{x} \wedge \frac{dy}{y}[r_x, r_y] \\ &= \sum_{x,y\in X\cap\pi} (\frac{dx}{x} \wedge \frac{dz^\pi}{z^\pi} + \frac{dz^\pi}{z^\pi} \wedge \frac{dy}{y})[r_x, r_y] \\ &= \sum_{x\in X\cap\pi} \frac{dx}{x} \wedge \frac{dz^\pi}{z^\pi}[r_x, \sum_{y\in X\cap\pi} r_y] - \sum_{y\in X\cap\pi} \frac{dy}{y} \wedge \frac{dz^\pi}{z^\pi}[\sum_{x\in X\cap\pi} r_x, r_y] \\ &= 0\end{aligned}$$

by the commutation relations. Using $\Omega = \sum_{\pi\in\Pi} \Omega^\pi$, we get that $\Omega = 0$. $\square$

Example 2.3. If $\mathcal{B}$ is two dimensional, we have an arrangement of lines in the plane, and then the condition is just

$$[r_j, \sum_{j'\in I} r_{j'}] = 0,$$

for any $r_j \in I$.

Definition 2.4. The holonomy Lie algebra $\mathfrak{a}(\mathcal{A})$ of the arrangement $\mathcal{A}$ is the quotient of the free Lie algebra generated by symbols r_i, $i \in I$, by the relations in Kohno's Lemma.

Thus, in other words, any linear representation $\pi : \mathfrak{a}(\mathcal{A}) \to \operatorname{End}\mathcal{F}$ of $\mathfrak{a}(\mathcal{A})$ is equivalent to a flat connection on $X \times \mathcal{F}$ of the form (1).

Since the relations satisfied by r_i are homogeneous, π gives rise to a one-parameter family of flat connections labeled by $h \in \mathbb{C}$, namely:

$$\nabla = d - h \sum_{i\in I} \frac{d\phi_i}{\phi_i} r_i,$$

and therefore to a one-parameter family of monodromy representations of the fundamental group $\pi_1(X)$ of X. These analytically deform the trivial representation of $\pi_1(X)$ on $\mathcal{F}$ which is obtained by setting $h = 0$.

2.1. Example: Knizhnik-Zamolodchikov (KZ) connection.

Let $\mathcal{B} = \mathbb{C}^n$ with coordinate $z_1, \ldots, z_n$. Take $\mathcal{A} = \{z_i = z_j\}_{i\neq j}$. So

$$X_n = \mathbb{C}^n \setminus \bigcup_{i\neq j} \{z_i = z_j\}$$

is the configuration space of n distinct ordered points in $\mathbb{C}$. We write:

$$X_n = \mathbb{C}^n \setminus \bigcup_{1\leq i<j\leq n} \Delta_{ij}$$

where $\Delta_{ij} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = z_j\}$, so that X_n is a hyperplane complement.

The connection we are considering is

$$\nabla = d - \sum_{i\neq j} \frac{d(z_i - z_j)}{z_i - z_j} r_{ij}, \tag{2}$$

Applying Lemma 2.1 to the above connection, we see that it is flat iff the following holds:

$$[r_{ij}, r_{jk} + r_{ik}] = 0, [r_{ij}, r_{kl}] = 0$$

for i, j, k, l all distinct.

To construct the Knizhnik-Zamolodchikov (KZ) connection on X_n, we fix a complex, semi-simple Lie algebra $\mathfrak{g}$ with a non-degenerate, invariant inner product $(,)$, one of its finite-dimensional representations V and set $\mathcal{F} = V^{\otimes n}$. The residue matrices r_{ij} are usually denoted by Ω_{ij} are given by

$$\Omega_{ij} = \sum_a \pi_i(X_a)\pi_j(X^a),$$

where $\pi_k(X)$ denotes the action of $X \in \mathfrak{g}$ on the kth tensor factor in $V^{\otimes n}$, and $\{X_a\}, \{X^a\}$ are dual basis of $\mathfrak{g}$.

A simple application of Kohno's lemma then shows that

$$\nabla_{KZ} := d - h \sum_{1\leq i<j\leq n} \frac{d(z_i - z_j)}{z_i - z_j}\Omega_{ij},$$

is a flat connection on $X_n \times V^{\otimes n}$ for any $h \in \mathbb{C}$. Check: the relations holds. To check that $[\Omega_{ij}, \Omega_{jk}+\Omega_{ik}] = 0$, it suffices to show that $i = 1, j = 2, k = 3$, note that $[\Omega, X\otimes 1+1\otimes X] = 0$, for any $X \in \mathfrak{g}$.

$$[\Omega \otimes 1, X_a \otimes 1 \otimes X^a + 1 \otimes X_a \otimes X^a] = [\Omega, X_a \otimes 1 + 1 \otimes X_a] \otimes X^a = 0$$

For distinct i, j, k, l, we have

$$\begin{aligned} \Omega_{ij}\Omega_{kl} &= \sum_a \sum_b \pi_i(X_a)\pi_j(X^a)\pi_k(X_b)\pi_l(X^b) \\ &= \sum_b \sum_a \pi_k(X_b)\pi_l(X^b)\pi_i(X_a)\pi_j(X^a) \\ &= \Omega_{kl}\Omega_{ij} \end{aligned}$$

Thus, $[\Omega_{ij}, \Omega_{kl}] = 0$.

Its monodromy yields one-parameter family of representations of Artin's pure braid group on n strands

$$P_n = \pi_1(\mathbb{C}^n \setminus \{z_i = z_j\}) \to \mathrm{GL}(V^{\otimes n})$$

which deforms the trivial representation of P_n on $V^{\otimes n}$.

We can however do a little better by noticing that the symmetric group S_n acts on $V^{\otimes n}$ and X_n. ∇_{KZ} is readily seen to be equivariant for the combination of these two actions and therefore descends to a flat connection on the quotient bundle $(X_n \times V^{\otimes n})/S_n$ over $\widetilde{X_n} = X_n/S_n$ i.e., the configuration space of n unordered points in $\mathbb{C}$. Taking its monodromy, we obtain a one-parameter family of representations of Artin's braid group on n strands:

$$\rho_h : B_n = \pi_1(\mathbb{C}^n \setminus \{z_i = z_j\}/S_n) \to \mathrm{GL}(V^{\otimes n})$$

ρ_h depends analytically in h and deforms the natural action of S_n on $V^{\otimes n}$, since ρ_0 factors through this action.

Recall that B_n is presented on elements T_i, $1 \leq i \leq n-1$, subject to Artin's braid relations

- $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, $i = 1, \dots, n-1$,
- $T_iT_j = T_jT_i$, $|i - j| \geq 2$.

Each T_i may be realized as a small loop in $\widetilde{X_n}$ around the image of the hyperplane $\{z_i = z_{i+1}\}$.

Example 2.5. Take $\mathfrak{g} = \mathfrak{gl}_m$ with the vector representation $V = \mathbb{C}^m$ and the inner product $\langle X, Y\rangle = \mathrm{tr}_V(XY)$. If $e_1, \dots, e_n$ is the standard basis of V and $E_{ij}e_k = \delta_{jk}e_i$ are the corresponding elementary matrices then, on $V^{\otimes 2}$,

$$\Omega_{12}(e_k \otimes e_l) = \sum_{1\le i,j\le m} E_{ij} \otimes E_{ji}(e_k \otimes e_l) = e_l \otimes e_k$$

so that Ω_{ij} acts on $V^{\otimes n}$ as the transposition (ij).

2.2. **Example: Coxeter-KZ connection.** Reference for this subsection, see: Page 161, of [14].

The connection described below was introduced by Cherednik in [2], to whom the results of this section are due, and is usually referred to as the KZ connection. In order to distinguish it from the one introduced in the previous subsection, we shall use the term Coxeter-KZ(CKZ) connection instead. Another name is Dunkl connection.

Example 2.6. From Example 2.5, the operator Ω_{12} acting on $V \otimes V$ is the same as the action of $(1, 2)$. Rewrite the KZ-connection in this case, we get:

$$\nabla = d - \sum_{i<j} h\frac{d(z_i - z_j)}{z_i - z_j}(i, j)$$

What about the other reflection groups?

Let W be a Weyl group (more genereally, W could be a finite Coxeter group, but trigonometric connections, to be defined in the next chapter, only exist for Weyl groups), with complexified reflection representation $\mathfrak{h} \cong \mathbb{C}^r$, and root system $R = \{\alpha\} \subset \mathfrak{h}^*$. The base space and arrangement are chosen by: $\mathcal{B} = \mathfrak{h}$ and $\mathcal{A} = \bigcup_{\alpha\in R} \mathrm{Ker}(\alpha)$, so that $X = \mathfrak{h}_{\mathrm{reg}}$ of regular elements in $\mathfrak{h}$. Set $\mathcal{F} = U$, where U is a finite dimensional W-module and let the residue r_α be given by the reflection $s_\alpha \in W$.

Theorem 2.7 (Cherednik)**.** *For any choice of weights $k_\alpha \in \mathbb{C}$ satisfying $k_{w\alpha} = k_\alpha$, for all $w \in W$, the connection*

$$\nabla_{CKZ} := d - \sum_{\alpha>0} k_\alpha \frac{d\alpha}{\alpha} s_\alpha,$$

is a W-equivariant, flat connection on $\mathfrak{h}_{\mathrm{reg}} \times U$.

Remark 2.8. The above connection is independent of the choice of a system of positive roots. Indeed, since $d \log a = d \log(-a)$, it may be rewritten as

$$\nabla_{CKZ} := d - \sum_{\alpha\in R} \frac{k_\alpha}{2}\frac{d\alpha}{\alpha} s_\alpha.$$

Proof. Let T be the set of reflections in W, thus, the set T is in bijection with the set of positive roots R^+. Let $S \subset T$ be a subset maximal with respect to the property in Kohno's lemma. Claim that: for any $s, t \in S$, we have $sts \in S$.

By Kohno's lemma, the flatness of ∇_{CKZ} is equivalent to the following commutation relation for every $S \subset T$ as above:

$$[s, \sum_{t\in S} k_t t] = 0,$$

for every $s \in S$. Now the left hand side of the required commutation relation can be written as:

$$[s, \sum_{t\in S} k_t t] = (s \sum_{t\in S} k_t t s^{-1} - \sum_{t\in S} k_t t)s$$

and let $s' = sts^{-1} \in S$ by the claim above, we get:

$$s \sum_{t\in S} k_t t s^{-1} = \sum_{t\in S} k_t sts^{-1} = \sum_{t\in S} k_{ss's^{-1}} s' = \sum_{s'\in S} k_{s'} s'$$

Thus,

$$[s, \sum_{t\in S} k_t t] = 0,$$

for every $s \in S$.

Next we prove W-equivariance of ∇_{CKZ}. Note that W acting on $\mathrm{End}(U)$ by conjugation. For any $s \in T$ be a reflection, let $H_s = \mathrm{Ker}(s-1)$ be the reflection hyperplane, denote α_s be the positive root that perpendicular to H_s. Thus, $s = s_{\alpha_s}$.

Moreover, $w\alpha_t$ is proportional to $\alpha_{wtw^{-1}}$. Combining these observations we have:

$$\begin{aligned} w^*\nabla_{CKZ} &= d - \sum_{t\in\Phi^+} k_t \frac{d(w.\alpha_t)}{w.\alpha_t} wtw^{-1} \\ &= d - \sum_{t\in\Phi^+} k_t \frac{d\alpha_{wtw^{-1}}}{\alpha_{wtw^{-1}}} wtw^{-1} \\ &= \nabla_{CKZ} \end{aligned}$$

again by using the fact that $k_t = k_{sts}$. □

The monodromy of ∇_{CKZ} yields a family of representation of the generalized pure braid group P_W of type W,

$$\rho_h : P_w = \pi_1(\mathfrak{h}_{\mathrm{reg}}) \to \mathrm{GL}(U)$$

Use the action of W on $\mathfrak{h}_{\mathrm{reg}}$ and U to push ∇_{CKZ} down to the quotient $\mathfrak{h}_{\mathrm{reg}}/W$. Since the connection is W-equivariant. This yields a representation of the generalized braid group:

$$\rho_h : B_w = \pi_1(\mathfrak{h}_{\mathrm{reg}}/W) \to \mathrm{GL}(U)$$

Note W being a Coxter group, there exist a choice of simple reflections $s_1, ..., s_r$, such that W is generated by $s_1, ..., s_r$, modulo the relations:

$$s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1$$

By Brieskorn's theorem [1], B_W is presented on generators $S_1, ..., S_r$ labelled by the same choice of simple reflections $s_1, ..., s_r$ in W with relations:

$$\underbrace{S_i S_j \ldots}_{m_{ij}} = \underbrace{S_j S_i \ldots}_{m_{ij}},$$

for any $1 \le i < j \le r$, where the number m_{ij} is equal to the order of $s_i s_j$ in W.

An explicit choice of representatives of $S_1, \ldots, S_n$ in $\pi_1(\mathfrak{h}_{\mathrm{reg}}/W)$ may be given as follows. See [16], Let $t \in \mathfrak{h}_{reg}$ lie in the fundamental Weyl chamber so that $\langle t, \alpha\rangle > 0$ for any $\alpha \in R^+$. Note that for any simple root α_i, the intersection $t_{\alpha_i} = t - \frac{1}{2}\langle t, \alpha_i\rangle\alpha_i^\vee$ of the affine line $t + \mathbb{C}\alpha_i^\vee$ with $\mathrm{Ker}(\alpha_i)$ does not lie in any other root hyperplane $\mathrm{Ker}(\beta)$, $\beta \in R \setminus \{\alpha_i\}$. Indeed, if $\langle t_{\alpha_i}, \beta\rangle = 0$, then

$$\langle t, \beta\rangle = \langle t, \alpha_i\rangle\langle \alpha_i^\vee, \beta\rangle = \langle t, \beta - s_i\beta\rangle$$

whence $\langle t, \beta\rangle = -\langle t, s_i\beta\rangle$, a contradiction since s_i permutes positive roots different from α_i. Let now D be an open disc in $t+\mathbb{C}\alpha_i^\vee$ of center t_{α_i} such that its closure D does not intersect any root hyperplane other than $\mathrm{Ker}(\alpha_i)$. Consider the path $\gamma_i : [0, 1] \to t + \mathbb{C}\alpha_i^\vee$ from t to $s_i t$ determined by $\gamma_i|_{[0,\frac{1}{3}]\cup[\frac{2}{3},1]}$ is affine and lies in $t + \mathbb{R}\alpha^\vee \setminus D$, $\gamma_i(\frac{1}{3}), \gamma_i(\frac{2}{3}) \in \partial\overline{D}$ and $\gamma_i|_{[\frac{1}{3},\frac{2}{3}]}$

is a semicircular arc in $\partial\overline{D}$, positively oriented with respect to the natural orientation of $t + \mathbb{C}\alpha_i^\vee$. Then, the image of γ_i in $\mathfrak{h}_{\mathrm{reg}}/W$ is a representative of S_i in $\pi_1(\mathfrak{h}_{\mathrm{reg}}/W, t)$. [1]

2.3. **Example: Casimir connection.** Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let $R = \{\alpha\} \subset \mathfrak{h}^*$ be the corresponding root system. The base space and arrangement are the same as those of the Coxeter-KZ connection for the Weyl group W of $\mathfrak{g}$, so that

$$X = \mathfrak{h} \setminus \bigcup_{\alpha\in R} \mathrm{Ker}(\alpha) = \mathfrak{h}_{\mathrm{reg}}$$

The fibre $\mathcal{F}$ of the vector bundle is now a finite-dimensional $\mathfrak{g}$-module U. To describe the residue matrices r_α, recall that for any root α, there is a corresponding subalgebra $\mathfrak{sl}_2^\alpha \subset \mathfrak{g}$ spanned by the triple $e_\alpha, f_\alpha, h_\alpha$, where $h_\alpha = \alpha^\vee \in \mathfrak{h}$ is the corresponding coroot and e_α, f_α are a choice of root vectors normalized by $[e_\alpha, f_\alpha] = h_\alpha$. The restriction of the inner product $\langle,\rangle$ of $\mathfrak{g}$ to $\mathfrak{sl}_2^\alpha$ determines a canonical Casimir element

$$C_\alpha = \frac{\langle\alpha,\alpha\rangle}{2}(e_\alpha f_\alpha + f_\alpha e_\alpha + \frac{1}{2}h_\alpha^2) \in U\mathfrak{sl}_2^\alpha \subset U\mathfrak{g}.$$

which we shall use as the residue on the hyperplane $\mathrm{Ker}(\alpha)$. The following result was discovered by De Concini around 1995(unpublished), and independently by J.Millson and V. Toledano Laredo [11], [16], see also [4]:

Theorem 2.9 (De Concini, Felder-Markov-Tarasov-Varchenko, Millson-Toledano Laredo)**.** *For any $h \in \mathbb{C}$, the Casimir connection*

$$\nabla_C = d - h\sum_{\alpha>0}\frac{d\alpha}{\alpha}C_\alpha$$

a flat connection on $\mathfrak{h}_{\mathrm{reg}} \times U$.

Proof. Again applying Kohno's lemma, we have to prove that for any two dimensional subspace $B \subset \mathfrak{h}^*$ spanned by a subset of Φ we have:

$$[C_\alpha, \sum_{\beta\in B} C_\beta] = 0$$

In order to show this let $\mathfrak{g}_B$ be subalgebra of $\mathfrak{g}$ corresponding to rank 2 system $B \cap \Phi$:

$$\mathfrak{g}_B := \bigoplus_{\alpha\in B\cap\Phi_+} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$

It is clear from the definitions that $\sum_{\beta\in B} C_\beta$ is same as the Casimir operator of $\mathfrak{g}_B$ modulo terms from $U\mathfrak{h}$. Hence we are done using the fact that each C_α commutes with elements from $U\mathfrak{h}$. □

We now wish to push the Casimir connection down to the quotient $\mathfrak{h}_{reg}/W$ to get a family of monodromy representations of the generalized braid group $B_\mathfrak{g} = B_W$. This requires a little work because the Weyl group W does not act on U and its Tits extension $\widetilde{W}$, while acting on U, does not act freely on $\mathfrak{h}_{\mathrm{reg}}$. For example, for $G = SL_2$, the generator of $W = \mathbb{Z}_2$ cannot be lifted to SL_2.

The reference of the following is [15], Appendix:

Definition 2.10. The Tits extension of W is the group $\widetilde{W}$ with generators $\widetilde{s_i}$, $i \in I$ and relations

- $\underbrace{\widetilde{s_i}\widetilde{s_j}\ldots}_{m_{ij}} = \underbrace{\widetilde{s_j}\widetilde{s_i}\ldots}_{m_{ij}}$,

- $\widetilde{s_i}^4 = 1$,
- $\widetilde{s_i}^2\widetilde{s_j}^2 = \widetilde{s_j}^2\widetilde{s_i}^2$,
- $\widetilde{s_i}\widetilde{s_j}^2\widetilde{s_i}^{-1} = \widetilde{s_j}^2(\widetilde{s_i}^2)^{-a_{ji}}$.

Recall that a representation V of $\mathfrak{g}$ is finite-dimensional representation if $\mathfrak{h} \subset \mathfrak{g}$ acts semi-simply with finite-dimensional eigenspaces and e_i, f_i act locally nilpotently.

Proposition 2.11. *Let V be an integrable representation of $\mathfrak{g}$. Then, the triple exponentials*

$$\exp(e_i)\exp(-f_i)\exp(e_i)$$

are well-defined elements of $\mathrm{GL}(V)$ and the assignment $\widetilde{s_i} \mapsto \exp(e_i)\exp(-f_i)\exp(e_i)$ yields a representation of V mapping $\widetilde{s_i}^2$ to $\exp(\pi\sqrt{-1}\alpha_i^\vee)$.

Proposition 2.12. *$\widetilde{W}$ is an extension of W by the abelian group Z generated by the elements $\widetilde{s_i}^2$. Z is isomorphic, as W-module to $Q^\vee/2Q^\vee \cong \mathbb{Z}_2^I$.*

We pullback the Casimir connection ∇_C to the universal cover

$$p : \widetilde{\mathfrak{h}_{\mathrm{reg}}} \to \mathfrak{h}_{\mathrm{reg}}.$$

From the presentation of $\widetilde{W}$, we know $\widetilde{W}$ is a quotient of $B_{\mathfrak{g}}$, the latter acts on U and, freely, on the universal cover $\widetilde{\mathfrak{h}_{\mathrm{reg}}}$. Consider the flat vector bundle $(\widetilde{\mathfrak{h}_{\mathrm{reg}}} \times U, p^*\nabla_C)/B_{\mathfrak{g}}$, note that the fundamental group of $\widetilde{\mathfrak{h}_{\mathrm{reg}}}/B_{\mathfrak{g}}$ is isomorphic to $B_{\mathfrak{g}}$. Thus, taking the monodromy of the flat vector bundle $(\widetilde{\mathfrak{h}_{\mathrm{reg}}} \times U, p^*\nabla_C)/B_{\mathfrak{g}}$ gives the desired one-parameter family ρ_h of representations of $B_{\mathfrak{g}}$.

3. Monodromy of rational connections

The following is a result we will need repeatedly. The statements can be found in [14], and the proofs of those statements can be found in [13].

Lemma 3.1. *Consider the ordinary differential equation*

$$\frac{df}{dz} = (\frac{A_0}{z} + \overline{A}(z))f, \tag{3}$$

where $\overline{A}(z) \in \mathcal{M}_n(\mathbb{C})$ is holomorphic in the neighborhood of 0. Assume that A_0 is non-resonant, that is the eigenvalues of A_0 do not differ by non-zero integers. Then, there exists a unique fundamental solution of the form

$$\Phi(z) = H(z)z^{A_0},$$

where $H(z)$ holomorphic near 0 and is normalized so as to have $H(0) = 1$.

Proof. We solve the system (3) formally, i.e, assume that a fundamental solution of required form exists with:

$$H(z) = H_0 + H_1 z + \cdots + H_k z^k + \dots$$

where $H_0 = 1$. Substituting $\Phi(z) = H(z)z^{A_0}$ in (3) we get

$$H' z^{A_0} + H z^{A_0} z^{-1} A_0 = z^{-1} A H z^{A_0},$$

which is equivalent to the following:

$$\sum_{m\geq 1} mH_m z^{m-1} + \sum_{m\geq 0} H_m A_0 z^{m-1} = \sum_{m\geq 0}(\sum_{r=0}^{m} A_r H_{m-r})z^{m-1}$$

Comparing the coefficients of z^m from both sides of the equation we get the following recursive system: Coefficient of z^{-1}: $H_0A_0 = A_0H_0$ holds by assumption that $H_0 = 1$. For every $m \geq 1$, we have:

$$mH_m + H_mA_0 = A_0H_m + A_1H_{m-1} + \cdots + A_{m-1}H_1 + A_m$$

which can be equivalently written as:

$$(m - \mathrm{ad}(A_0))H_m = A_1H_{m-1} + \cdots + A_{m-1}H_1 + A_m$$

where $\mathrm{ad}(A_0)$ is operator on $\mathcal{M}_n(\mathbb{C})$ defined as: $X \mapsto A_0X - XA_0$. The assumption that the eigenvalues of A_0 do not differ by non-zero integers implies the operator $m - ad(A_0)$ is an invertible operator on $\mathcal{M}_n(\mathbb{C})$ for each $m \geq 1$, thus, the above system has a unique solution. □

3.1. **Hecke algebras and monodromy representation of CKZ-connections.** Recall by Brieskorn's theorem , B_W is presented on generators $S_1, ..., S_r$ labelled by a choice of simple reflections $s_1, ..., s_r$ in W with relations:

$$\underbrace{S_iS_j\ldots}_{m_{ij}} = \underbrace{S_jS_i\ldots}_{m_{ij}},$$

for any $1 \leq i < j \leq r$, where the number m_{ij} is equal to the order of s_is_j in W.

Definition 3.2. Given invertible elements v_i of a ring R, such that $v_i = v_j$ whenever the reflections s_i and s_j are conjugate in W, the Hecke algebra $H_W(v_i)$ of W is the quotient of the group algebra RB_W by the relations

$$(S_i - v_i)(S_i + v_i^{-1}) = 0.$$

In particular, when $v_i = 1$, the Hecke algebra $H_W(1) = \mathbb{C}W$.

Proposition 3.3. *Assume $k_i \notin \frac{1}{2}\mathbb{Z}$, then, the monodromy of ∇_{CKZ} over $\mathfrak{h}_{\mathrm{reg}}$ around* $\mathrm{Ker}(\alpha)$ *is conjugate to $e^{2\pi\sqrt{-1}k_{\alpha_i}s_{\alpha_i}}$.*

Proof. Consider an affine $\mathbb{C}-$plane π_α, complementary to $\mathrm{Ker}(\alpha)$, that is, π_α is given by $x_0 + z\alpha^\vee$, for $z \in \mathbb{C}$, and $x_0 \in \mathrm{Ker}(\alpha) \setminus \cup_{\beta\neq\alpha} \mathrm{Ker}(\beta)$. The loop γ_α is described by $x_0 + e^{2\pi\sqrt{-1}t}\alpha^\vee$, for $t \in [0, 1]$.

Now restrict the connection ∇_{CKZ} to π_α, we get

$$\begin{aligned}\nabla_{CKZ}|_{\pi_\alpha} &= d - \sum_{\beta\in\Phi^+} k_\beta \frac{\beta(\alpha^\vee)}{\beta(x_0 + z\alpha^\vee)} dz \\ &= d - k_\alpha s_\alpha \frac{dz}{z} - \sum_{\beta\neq\alpha, \beta(\alpha^\vee)\neq 0} k_\beta \frac{dz}{z + \frac{\beta(x_0)}{\beta(\alpha^\vee)}},\end{aligned}$$

where $\frac{\beta(x_0)}{\beta(\alpha^\vee)} \neq 0$.

Now the connection $\nabla_{CKZ}|_{\pi_\alpha}$ has the form (3), with $A_0 = k_\alpha s_\alpha$, which is non-resonant precisely when $k_\alpha \notin \frac{1}{2}\mathbb{Z}$. Using the lemma above, we get a fundamental solution $\Phi(z) = H(z)z^{k_\alpha s_\alpha}$.

Thus, monodromy around $\ker(\alpha)$ is $\mu_\Phi(\gamma) = \Phi(\gamma(0))^{-1}\Phi(\gamma(1)) = e^{2\pi\sqrt{-1}k_\alpha s_\alpha}$. □

Proposition 3.4. *Assume $k_i \notin \frac{1}{2}\mathbb{Z}$, the monodromy of ∇_{CKZ} over $\mathfrak{h}_{\mathrm{reg}}/W$ around* $\mathrm{Ker}(\alpha)$ *is conjugate to $s_ie^{\pi\sqrt{-1}k_{\alpha_i}s_{\alpha_i}}$, for any $i \in I$.*

Proof. The proof is similar as the proof of above Proposition 3.3. First the $\mathbb{C}$-plane π_i in the proof of Proposition 3.3 is invariant under s_i. Since $\pi_i = x_0 + \mathbb{C}\alpha_i^\vee$, and $x_0 \in \operatorname{Ker}(\alpha_i)$, thus, the action of s_i simply corresponds to the negation $z \mapsto -z$.

Now we have a connection on $\mathbb{C}$ of the form $d - A(z)dz$, which is equivariant under the action of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, let σ be an automorphism by which the generator of $\mathbb{Z}_2$ acts on the fiber U of ∇_{CKZ}. then, we have:

$$A(-z) = -\operatorname{Ad}(\sigma)A(z) \tag{4}$$

Assume now that $A(z) = \frac{A_0}{z} + \overline{A}(z)$, and note that (4) implies that $\operatorname{Ad}(\sigma)A_0 = A_0$.

If A_0 is non-resonant, we have a unique canonical solution $\Psi(z) = H(z)z^{A_0}$ in the neighborhood of $z = 0$. The equivariance implies that

$$\operatorname{Ad}(\sigma)(H(-z)) = H(z)$$

Let $z_0 \neq 0$ be a base point in $\mathbb{C}$. Now the monodromy along the half loop $\gamma = z_0 e^{\pi i t}$, where $0 \leq t \leq 1$, is

$$\Psi(z_0)^{-1}\sigma\Psi(-z_0),$$

where σ is used to identify the fiber of the vector bundle at z_0 and the fiber at $-z_0$.

$$\begin{aligned}
&z_0^{-A_0}H(z_0)^{-1}\sigma H(-z_0)(-z_0)^{A_0}\\
=&z_0^{-A_0}H(z_0)^{-1}\operatorname{Ad}(\sigma)(H(-z_0))(-z_0)^{A_0}\sigma\\
=&z_0^{-A_0}(-z_0)^{A_0}\sigma\\
=&\exp(i\pi A_0)\sigma\\
=&\sigma\exp(i\pi A_0)\text{since } \sigma \text{ commutes with } A_0.
\end{aligned}$$

Note that here we get $\exp(i\pi A_0)$ instead of $\exp(-i\pi A_0)$ is because we are taking the monodromy along the half loop $\gamma = z_0 e^{\pi i t}$, where $0 \leq t \leq 1$. □

Remark 3.5. It's not true that one can simultaneously conjugate all the S_i to the corresponding $s_i e^{\pi\sqrt{-1}k_{\alpha_i}s_{\alpha_i}}$.

Each simple reflection s_{α_i} has two eigenvalues ± 1 in U, which implies the operator $\mu_k(S_i)$ is semisimple, with eigenvalues $\pm e^{\pm\pi\sqrt{-1}k_{\alpha_i}} = \pm v_i^{\pm 1}$. Thus, the quadratic relations holds:

$$(S_i - v_i)(S_i + v_i^{-1}) = 0,$$

for all generic k_i. Since the S_i vary continuously (in fact analytically) in k_i, the relation $(S_i - v_i)(S_i + v_i^{-1}) = 0$ must hold for all k_i.

That is, monodromy of ∇_{CKZ} factors through the Hecke algebra $H_W(v_i)$, with $v_i = e^{\pi\sqrt{-1}k_{\alpha_i}}$.

$$\begin{tikzcd}
\mathbb{C}B_W \arrow[rr] \arrow[dr] & & \operatorname{End}(U) \\
& H_W(v_i) \arrow[ur] &
\end{tikzcd} \tag{5}$$

Choosing U to be the direct sum of the irreducible representations of W, so that $\operatorname{End}(U) \cong \mathbb{C}W$, and the weights k_α to be generic, the monodromy does in fact yield an algebra isomorphism of $H_W(v_i) \cong \mathbb{C}W$.

One way to see the isomorphism is, if working over $C[[k_i]]$, then (by a Theorem of Tits) H_W is a flat deformation of $\mathbb{C}W$. Now we have a map $H_W \to C[[k_i]]W$ which is an isomorphism mod k_i and therefore is an isomorphism.

For numerical k_i, by the same result of Tits that $\dim H_W = \dim \mathbb{C}W$ for v_i are nonzero, then we have a family of monodromy maps labeled by nonzero v_i between two vector spaces of the same dimension. Since the family is an isomorphism at $v_i = 1$, it is an isomorphism generically.

3.2. **Monodromy of KZ-connection.** The Drindeld-Jimbo quantum group $U_\hbar(\mathfrak{g})$ is a topological Hopf algebra, which is a deformation of the enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$, i.e., a Hopf algebra over the ring $\mathbb{C}[[\hbar]]$ of formal power series in the variable $\hbar$, which is topologically free as $\mathbb{C}[[\hbar]]$-module and endowed with an isomorphism $U_\hbar(\mathfrak{g})/\hbar U_\hbar(\mathfrak{g}) \cong U\mathfrak{g}$ of Hopf algebras.

Definition 3.6. Let $\mathfrak{g}$ be a semisimple Lie algebra, with Cartan matrix $A = (a_{ij})$, the quantum group $U_\hbar(\mathfrak{g})$ is generated by $\{E_i, F_i, H_i\}_{1\le i\le n}$, subject to the following relations:

- $[H_i, H_j] = 0, [H_i, E_j] = a_{ij}E_j, [H_i, F_j] = -a_{ij}F_j$;
- $[E_i, F_j] = \delta_{ij}\frac{e^{\hbar d_i H_i/2} - e^{-\hbar d_i H_i/2}}{e^{\hbar d_i/2} - e^{-\hbar d_i/2}}$,
- For $i \neq j$,
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0$$
and
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0$$

where $q_i = e^{\hbar d_i/2}$, and where
$$\sinh x = \frac{e^x - e^{-x}}{2},$$
and
$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, [n]_q! = [n]_q[n-1]_q \cdots [1]_q$$
are the usual q-numbers and factorials.

Remark 3.7. Note that $\mathrm{ad}(x)^m(y) = \sum_{k=0}^m (-1)^k \binom{m}{k} x^k y x^{m-k}$, thus the relation in $U_\hbar(\mathfrak{g})$ is a deformation of Serre's relation. And we also have:
$$\frac{\sinh(\hbar d_i H_i/2)}{\hbar d_i/2} \equiv H_i \mod \hbar$$

Example 3.8. Assume $\mathfrak{g} = \mathfrak{sl}_2$, $U_\hbar(\mathfrak{sl}_2)$ is generated by E, F, H, subject to the following relations:
$$[H, E] = 2E, [H, F] = -2F,$$
and
$$[E, F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^\hbar - e^{-\hbar}}$$

Any finite dimensional representation $\mathcal{V}$ of $U_\hbar(\mathfrak{g})$, i.e., one which is finitely generated and topologically free as $\mathbb{C}[[\hbar]]$-module, is uniquely determined by the $\mathfrak{g}$-module $V = \mathcal{V}/\hbar\mathcal{V}$. Indeed, since $H^2(\mathfrak{g}, U\mathfrak{g}) = 0$, the multiplication in $U\mathfrak{g}$ does not possess non-trivial deformations and $U_\hbar(\mathfrak{g})$ is isomorphic as $\mathbb{C}[[\hbar]]$-algebra to $U\mathfrak{g}[[\hbar]]$.

Using this to let $U\mathfrak{g}$ act on $\mathcal{V}$, we may regard the latter as a deformation of V. Since $H^1(\mathfrak{g}, End(V)) = 0$ however, $\mathcal{V}$ is isomorphic, as $U\mathfrak{g}$ and therefore as $U_\hbar(\mathfrak{g})$-module, to the trivial deformation $V[[\hbar]]$ of V .

Theorem 3.9 (Faddeev-Reshetikhin-Takhtajan, Drinfeld, Jimbo)**.** *There exists a universal R-matrix $R \in U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{g})$, such that the elements $R_1^\vee, \dots, R_n^\vee \in \mathrm{GL}(\mathcal{V}^{\otimes n})$ given by*

$$R_i^\vee = (i, i+1) \cdot 1 \otimes \cdots \otimes 1 \otimes R \otimes 1 \otimes \cdots \otimes 1$$

commute with $U_\hbar(\mathfrak{g})$ and satisfy

- *the braid relations*
  (1): $R_i^\vee R_{i+1}^\vee R_i^\vee = R_{i+1}^\vee R_i^\vee R_{i+1}^\vee$, $i = 1, \dots, n$.
  (2): $R_i^\vee R_j^\vee = R_j^\vee R_i^\vee$, $|i - j| \geq 2$.
- *the deformation property: $R_i^\vee = (i, i+1) + o(\hbar)$.*

From above theorem, we have a map $B_n \to S_n \rtimes (U_\hbar(\mathfrak{g}))^{\otimes n}$, $T_i \mapsto R_i^\vee$. Thus, B_n acts on $\mathcal{V}^{\otimes n}$ by R-matrix representation of $U_\hbar(\mathfrak{g})$.

$$\begin{tikzcd}
 & \mathrm{GL}(V \otimes \cdots \otimes V[[\hbar]]) \\
B_n \arrow[ur, "\nabla_{KZ}"] \arrow[dr, "R"'] & \\
 & \mathrm{GL}(\mathcal{V} \otimes \cdots \otimes \mathcal{V})
\end{tikzcd}$$

The statement of the following theorem can be found in [14], page 166. See [3] for the proof.

Theorem 3.10 (Kohno, Drinfeld)**.** *The monodromy representation of the KZ equations on $V^{\otimes n}[[\hbar]]$ is equivalent to the R-matrix action of B_n on $\mathcal{V}^{\otimes n}$.*

3.3. **Monodromy Representations of Casimir-connection.** Before stating the precise result, recall that the latter action arises by mapping $\widetilde{W}$ to the completion $\widehat{U(\mathfrak{g})}$ of $U\mathfrak{g}$ with respect to its finite-dimensional representations via

$$\widetilde{s_i} \mapsto \exp(e_i) \exp(-f_i) \exp(e_i)$$

Let $q_i = e^{\hbar \frac{\langle \alpha_i, \alpha_i \rangle}{2}}$ and consider the triple q-exponentials

$$S_i = \exp_{q_i^{-1}}(q_i^{-1} E_i q_i^{-H_i}) \exp_{q_i^{-1}}(-F_i) \exp_{q_i^{-1}}(q_i E_i q_i^{H_i}),$$

where E_i, F_i, H_i are the generators of the subalgebra $U_\hbar \mathfrak{sl}_2^i \subset U_\hbar \mathfrak{g}$ corresponding to the simple root α_i,

$$\exp_q(x) = \sum_{n \geq 0} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n_q]!},$$

Viewing the S_i as lying in the completion $\widehat{U_\hbar \mathfrak{g}}$ of $U_\hbar \mathfrak{g}$ with respect to its finite-dimensional representations, we have the following

Theorem 3.11 (Lusztig, Kirillov-Reshetikhin, Soibelman)**.** *The elements $S_1, \dots, S_r$ satisfy*

- *the braid relations*
$$S_i S_j S_i \cdots = S_j S_i S_j \cdots$$
where there are m_{ij} factors on each side.

- *the deformation property:* $S_i = s_i + o(\hbar)$.

The quantum Weyl group action is given by the S_i. Just as the operators $R_i^\vee$, each S_i is local in that it lies in the completion $\widehat{U_\hbar \mathfrak{sl}_2^i}$ of $U_\hbar \mathfrak{sl}_2^i$, and does not square to 1.

$$\begin{tikzcd}
 & \mathrm{GL}(V\otimes\cdots\otimes V[[\hbar]]) \\
B_{\mathfrak{g}} \arrow[ur, "\nabla_C"] \arrow[dr, "qW"'] & \\
 & \mathrm{GL}(\mathcal{V}\otimes\cdots\otimes\mathcal{V})
\end{tikzcd}$$

The statement of the following theorem can be found in [14], and proved in [16] for $\mathfrak{g} = \mathfrak{sl}_n$.

Theorem 3.12 (V.Toledano Laredo)**.** *The monodromy of the Casimir connection ∇_C with values in $V[[\hbar]]$ is equivalent to the quantum Weyl group action of $B_{\mathfrak{g}}$ on $\mathcal{V}$.*

The quantum Weyl group action deforms the Tits extension $\widetilde{W}$ on finite-dimensional $\mathfrak{g}$-modules.

Appendix A. Flat connections and monodromy representations

The reference for this part is [9]

A.1. **Flat connections over principal bundles.** Let $P(M, G)$ be a principal G-bundle over a manifold M with group G. For each $u \in P$, let T_uP be the tangent space of P at u and V_u the subspaces of T_uP consisting of vectors tangent to the fiber through u.

Definition A.1. A connection ∇ in P is an assignment of a subspace H_u of T_uP to each $u \in P$, such that

- **(a):** $T_uP = V_u \oplus H_u$;
- **(b):** $Q_{ug} = (R_g)_* H_u$, for every $u \in P$, and $g \in G$, where R_g is the transformation of P induced by $g \in G$, $R_g u = ug$;
- **(c):** H_u depends differentiably on u.

Denote $\pi : P \to M$ be the projection. From above definition, the connection ∇ gives a splitting of $D\pi$, that is,

$$\begin{tikzcd}
0 \arrow[r] & T_{\pi(u)}M \arrow[r, "\nabla"] \arrow[dr, equal] & T_uP \arrow[d, "D\pi"] \\
 & & T_{\pi(u)}M
\end{tikzcd}$$

The decomposition $T_uP = V_u \oplus H_u = \mathrm{Ker}(D\pi) \oplus \mathrm{im}(\nabla)$.

An equivalent definition is the following:

Definition A.2. A principal G-connection form on P is a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ on P with values in the Lie algebra $\mathfrak{g}$ of G, such that,

- **(a):** $\omega(A^*) = A$, for any $A \in \mathfrak{g}$, where A^* is a vector field on P induced by $A \in \mathfrak{g}$.
- **(b):** $(R_g)^*\omega = \mathrm{ad}(g^{-1})\omega$, for any $g \in G$, where ad denotes the adjoint representation of G in $\mathfrak{g}$.

Given a connection ∇ in P, we define a 1-form ω as follows: For any $X \in T_uP$, we define $\omega(X)$ to be the unique $A \in \mathfrak{g}$, such that $(A^*)_u$ is equal to the vertical component of X. It's clear that $\omega(X) = 0$ if and only if X is horizonal.

Conversely, given a form ω, we define

$$H_u := \{X \in T_uP \mid \omega(X) = 0\}.$$

Definition A.3. The curvature form of a principal G-connection ω is the $\mathfrak{g}$-valued 2-form Ω defined by

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega].$$

A connection is called *flat* if the curvature form vanishes identically.

Given a piecewise differentiable curve $\tau = x_t, 0 \leq t \leq 1$ in M. A *horizontal lift* of τ is a horizontal curve $\tau^* = u_t, 0 \leq t \leq 1$ in P, such that $\pi(u_t) = x_t$, for $0 \leq t \leq 1$. Here horizonal curve means whose tangent vectors are all horizontal.

Proposition A.4. *Notations as above, for any arbitrary point u_0 of P with $\pi(u_0) = x_0$, there exists a unique lift $\tau^* = u_0$, which starts from u_0.*

Now using above proposition, we define the parallel displacement of fibres as follows. Let u_0 be an arbitrary point of P, with $\pi(u_0) = x_0$. The unique lift τ^* of τ starting at u_0 has the end point u_1, such that $\pi(u_1) = x_1$.

By varying u_0 in the fiber $\pi^{-1}(x_0)$, we obtain a mapping of the fiber $\pi^{-1}(x_0)$ onto the fiber $\pi^{-1}(x_1)$, which maps u_0 to u_1. We call this mapping the *parallel displacement* along the curve τ.

Note $\tau : \pi^{-1}(x_0) \to \pi^{-1}(x_1)$ is actually an isomorphism, since:

Proposition A.5. *The parallel displacement along any curve τ commutes with the action of G on P:*

$$\tau \circ R_g = R_g \circ \tau,$$

for every $g \in G$.

Theorem A.6. *If the connection ∇ is flat, then the parallel displacement $\pi^{-1}(x_0) \to \pi^{-1}(x_1)$ is unchanged by homotopies.*

Thus, take $x_0 = x_1 \in M$, the parallel displacement gives a map

$$\pi_1(M, x_0) \to G,$$

which we call it *monodromy* of the flat connection ∇.

A.2. **Flat connections over vector bundles.** Let V be a vector bundle on base space M.

Definition A.7. A connection is a $\mathbb{C}$-linear map

$$\nabla : V \to \Omega^1_M \otimes_{\mathscr{O}_M} V,$$

which satisfies the Leibniz rule, that is, for local sections f of $\mathscr{O}_M$, and s of V, we have

$$\nabla(fs) = df \otimes s + f\nabla(s).$$

A connection ∇ is *flat* if the curvature $(\nabla \circ \nabla)$ of the connection is zero.

Example A.8. Recall that local systems are locally constant sheaves of finite rank. Let $\mathbb{V}$ be a $\mathbb{C}$-local system on M. Then, there is a canonical connection associated with $\mathbb{V}$ on the vector bundle $\mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_M$. Let U be an open set, where $\mathbb{V}$ can be trivialized, and let $s_1, \ldots, s_n$ be the basis of $\mathbb{V}(U)$ given by the trivialization. For a section $s = \sum_i f_i s_i \in V(U)$, define

$$\nabla(s) = \sum_i df_i \otimes s_i.$$

The local definition is compatible with the coordinate changes, as they are given by locally constant matrices, hence it gives rise to a global map ∇. It follows from the definition that ∇ is $\mathbb{C}$-linear and satisfies the Leibniz rule, so ∇ is a connection on V.

In particular, we have $\nabla(s_i) = 0$, for all $i = 1, \ldots, n$, and even $\mathrm{Ker}(\nabla) = \mathbb{V}$. For locally on a coordinate neighborhood U of M, where $\mathbb{V}$ can be trivialized, we can write the image $\nabla(s)$ of $a \in V(U)$ as

$$\nabla(s) = \sum_i df_i \otimes s_i = \sum_{ij} \frac{\partial f_i}{\partial z_i} dz_j \otimes s_i$$

and the set $\{dz_j \otimes s_i\}_{i,j}$ is a $\mathscr{O}_M(U)$-basis of $\Omega^1_M \otimes_{\mathscr{O}_M} V(U)$. Therefore, $\nabla(s) = 0$ implies $\frac{\partial f_i}{\partial z_i} = 0$, and the f_i must be locally constant functions.

Proposition A.9. *The functor*

$$\mathbb{V} \mapsto (\mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_M, \nabla_{\mathbb{V}}),$$

from the category of $\mathbb{C}$-local systems on M to the category of vector bundles on M equipped with a flat connection has a quasi-inverse

$$(V, \nabla) \mapsto \mathrm{Ker}(\nabla).$$

Proposition A.10. *Let M be a path-connected, locally simply connected topological space with base point x. Then there is an equivalence between the category of $\mathbb{C}$-local systems on M and the category of $\pi_1(X, x)$-left modules, given by the functor:*

$$\mathbb{V} \to \mathbb{V}_x$$

Proof. Given a path $c : [0,1] \to M$, starting at $x = c(0)$, there is a unique way of continuing a germ $v \in \mathbb{V}_x$ along γ to an element $v' \in \mathbb{V}_{c(1)}$(since every germ in $\mathbb{V}_x$ produces a unique section $\mathbb{V}(U)$ for some neighborhood U of x). This continuation process only depends on the homotopy class. Thus, it allows us to define a representation

$$\pi_1(M, x) \to \mathrm{Aut}(\mathbb{V}_x).$$

For the inverse functor, we start with a representation

$$\rho : \pi_1(M, x) \to \mathrm{Aut}(V)$$

and consider the constant sheaf $V_{\widetilde{M}}$ on the universal cover $u : \widetilde{M} \to X$. We define an $\mathbb{C}$ local system $\mathbb{V}$ on M by taking on an open set $U \subset X$ the section $f : u^{-1}(U) \to V$ of $V_{\widetilde{M}}$ that satisfy:

$$f(\gamma x) = \rho(\gamma) f(x)$$

for all $\gamma \in \pi_1(M, x), x \in u^{-1}(U)$. Then $\mathbb{V}$ is isomorphic to the constant sheaf V_U on a sufficiently small neighborhood $U \subset M$, so it is a local system. ◻

Thus, we start with a vector bundle V on M, with fiber F, then the monodromy of a flat connection ∇ on the vector bundle V gives a representation of $\pi_1(M, x_0)$:

$$\pi_1(M, x_0) \to \mathrm{Aut}(F).$$

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