QUANTUM HAMILTONIAN REDUCTION

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1. The quantization formalism

1.1. **Preliminaries and notation.** Throughout, we work over $\mathbb{C}$. Assume that $\mathfrak{g}$ is the reductive Lie algebra associated to a reductive algebraic group G.

1.2. **Three notions of quantization.** Throughout this section, let A_0 be a commutative Poisson algebra. We describe the concepts of formal, graded, and filtered quantization of A_0 and how to translate between them under different conditions.

(a) A *formal quantization* of A_0 is a $\mathbb{C}[[\hbar]]$-algebra A, flat over $\mathbb{C}[[\hbar]]$ and complete and separated in the $\hbar$-adic topology, so that $A/\hbar A \simeq A_0$ and $\{a,b\} = \frac{1}{\hbar}[\overline{a},\overline{b}] \mod \hbar$. In this case, we also say that $(A, \{-,-\})$ is the *quasi-classical limit* of A.

(b) Suppose that A_0 is $\mathbb{Z}_{\geq 0}$-graded so that the Poisson bracket has degree -1. A *graded quantization* (also called *algebraic quantization*) of A_0 over $\mathbb{C}[\hbar]$ is a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}[\hbar]$-algebra $A_\hbar$ so that $\deg(\hbar) = 1$ which is free as a $\mathbb{C}[\hbar]$-module together with an isomorphism of graded algebras $A_\hbar/\hbar A_\hbar \to A_0$ such that $\{a,b\} = \frac{1}{\hbar}[\overline{a},\overline{b}] \mod \hbar$.

(c) Suppose that A_0 is $\mathbb{Z}_{\geq 0}$-graded so that the Poisson bracket has degree -1. A *filtered quantization* of A_0 is a filtered algebra A where $[-,-]$ decreases the degree by 1 so that $\mathrm{gr}A = A_0$ and $\mathrm{gr}([\overline{a},\overline{b}]) = \mathrm{gr}(\{a,b\})$.

Suppose further that A_0 is $\mathbb{Z}_{\geq 0}$-graded and that $\deg(\{-,-\}) = -1$. Given a formal quantization $A_\hbar$ of A_0 equipped with a $\mathbb{C}^*$-action on $A_\hbar$ inducing the $\mathbb{Z}_{\geq 0}$-grading on A_0, the subalgebra $A_{\hbar,\mathrm{fin}} \subset A_\hbar$ of vectors with locally finite action of $\mathbb{C}^*$ is a graded quantization of A_0. Conversely, given a graded quantization $A_{\hbar,\mathrm{fin}}$, we may form the corresponding formal quantization $A_\hbar := \widehat{A}_{\hbar,\mathrm{fin}}$ as its $\hbar$-adic completion.

By Propositions 1.1 and 1.2, we may convert between filtered and graded quantizations. For a filtered quantization A_f of A_0, the construction proceeds via the Rees algebra

$$\mathrm{Rees}(A_f) = \bigoplus_{n \geq 0} A_f^n \cdot \hbar^n \subset A_f[\hbar].$$

Proposition 1.1. If A_f is a filtered quantization of A_0, $A_\hbar := \mathrm{Rees}(A_f)$ viewed as a graded $\mathbb{C}[\hbar]$-subalgebra of $A_f[\hbar]$, is a graded quantization of A_0.

Proof. By construction, we see that $A_\hbar/\hbar A_\hbar \simeq \mathrm{gr}A_f = A_0$. Further, for elements $a^i \in A_f^i$ and $a^j \in A_f^j$, because A_f is a filtered quantization of A_0, we have $[a^i, a^j] \in A_f^{i+j-1}$ and $\mathrm{gr}([a^i,a^j]) = \mathrm{gr}(\{a^i,a^j\})$. Therefore, in $A_\hbar$, we have $[a^i,a^j] \in \hbar A_\hbar^{i+j-1}$, so that $\frac{1}{\hbar}[a^i,a^j] = \{a^i,a^j\} \mod \hbar$. □

Proposition 1.2. Suppose that $A_\hbar$ is a graded quantization of A_0 which is $\mathbb{Z}_{\geq 0}$-graded so that $\deg(\hbar) = 1$. Then $A_f := A_\hbar/(\hbar - 1)A_\hbar$ with filtration induced from the grading of $A_\hbar$ is a filtered quantization of A_0.

Proof. Denote the graded parts of $A_\hbar$ by $A_\hbar^n$ and the corresponding filtration by $A_\hbar^{\leq n}$. Because the element $\hbar$ has degree 1, we find that

$$\mathrm{gr}A_\hbar = \bigoplus_{n \geq 0} (A_\hbar^{\leq n}/A_\hbar^{\leq n-1})/(\hbar - 1)(A_\hbar^{\leq n-1}/A_\hbar^{\leq n-2}) = \bigoplus_{n \geq 0} A_\hbar^n/\hbar A_\hbar^{n-1} = A_\hbar/\hbar A_\hbar.$$

Further, for $a^i \in A_\hbar^i$ and $a^j \in A_\hbar^j$, we see that

$$[a^i,a^j] = \{a^i,a^j\} \pmod{\hbar} \text{ in } A_\hbar^{i+j}/(\hbar - 1)A_\hbar^{i+j-1},$$

which shows exactly that $\mathrm{gr}([a^i,a^j]) = \mathrm{gr}(\{a^i,a^j\})$. □

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1.3. **Two homogenized algebras.** We now give two examples of graded quantizations which will be important in the next section. Define the *homogenized universal enveloping algebra* of a Lie algebra $\mathfrak{g}$ to be the $\mathbb{C}[\hbar]$-algebra

$$U_\hbar(\mathfrak{g}) := \Big\langle \hbar, x \in \mathfrak{g} \mid x \star y - y \star x = \hbar[x, y] \Big\rangle,$$

where $\star$ denotes the product in $U_\hbar(\mathfrak{g})$. Notice that $U_\hbar(\mathfrak{g})$ is a graded quantization of $S(\mathfrak{g})$ and that the corresponding filtered quantization $U_\hbar(\mathfrak{g})/(\hbar - 1)U_\hbar(\mathfrak{g}) \simeq U(\mathfrak{g})$ is the standard universal enveloping algebra. The corresponding Poisson algebra structure on $S(\mathfrak{g})$ is the one induced by $\{x, y\} = [x, y]$.

Let X be a smooth affine algebraic variety. Define the ring of *homogenized differential operators on* X to be the $\mathbb{C}[\hbar]$-algebra generated by $\mathcal{O}_X$ in grade 0, $\mathrm{Vect}(X)$ in grade 1 and subject to the relations

$$D_\hbar(X) := \langle f, v \mid f \star g = f \cdot g, f \star v = f \cdot v, v \star f = f \cdot v + \hbar v(f), u \star v - v \star u = \hbar[u, v]\rangle\,,$$

where we use $\star$ to denote the product in $D_\hbar(X)$. Observe that $D_\hbar(X)$ is a graded quantization of $p_*\mathcal{O}_{T^*X}$ (where $p : T^*X \to X$ is the natural projection) and that $D_\hbar(X)/(\hbar - 1)D_\hbar(X) = D(X)$, the ordinary ring of differential operators on X.

2. Quantum reduction formalism

We now define quantum reduction, first in the general case and then in the case of a graded quantization.

2.1. **Quantum moment map.** Let A be an associative algebra equipped with a locally-finite completely reducible $\mathfrak{g}$-action given by a map of Lie algebras $\phi : \mathfrak{g} \to \mathrm{Der}(A)$. We say that $\mu : U(\mathfrak{g}) \to A$ is a *quantum moment map* for ϕ if $\phi(x)(-) = [\mu(x), -]$ so that the action of $\mathfrak{g}$ factors through μ.

Example. Let X be an affine algebraic variety with an action of G. Differentiation of the group action gives rise to a Lie algebra map $\mu : \mathfrak{g} \to D(X)$, where $D(X)$ is the algebra of differential operators on X.

In the homogenized context, if $A_\hbar$ is a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}[\hbar]$-algebra so that $\deg(\hbar) = 1$ with grading-preserving $\mathfrak{g}$-action $\phi : U_\hbar(\mathfrak{g}) \to \mathrm{Der}(A_\hbar)$, we say that $\mu_\hbar : U_\hbar(\mathfrak{g}) \to A_\hbar$ is a *homogenized quantum moment map* if the action is given by $\phi(x) = \frac{1}{\hbar}[\mu_\hbar(x), -]$ and $\mu_\hbar(\mathfrak{g}) \subset A^1_\hbar$. In the previous example, we may obtain a *homogenized quantum moment map* $\mu_\hbar : U_\hbar(\mathfrak{g}) \to D_\hbar(X)$ in the same way. Observe that $\mu = \mu_\hbar \pmod{\hbar - 1}$.

2.2. **Definition of quantum reduction.** Suppose now that we have either a quantum moment map μ or homogenized quantum moment map $\mu_\hbar$. For a character $\lambda : \mathfrak{g} \to \mathbb{C}$, let $J_{\mu,\lambda} := A \cdot \mathrm{span}\{\mu(x) - \langle\lambda, x\rangle \mid x \in \mathfrak{g}\}$ (in the graded context $J_{\mu,\lambda,\hbar} := A_\hbar \cdot \mathrm{span}\{\mu_\hbar(x) - \hbar\langle\lambda, x\rangle \mid x \in \mathfrak{g}\}$, a graded ideal). Observe that the $\mathfrak{g}$-action fixes $J_{\mu,\lambda}$ and $J_{\mu,\lambda,\hbar}$. We define the *quantum Hamiltonian reduction* of A with respect to $\lambda : \mathfrak{g} \to \mathbb{C}$ and μ to be the algebra

$$R(\mathfrak{g}, A, \lambda) := \mathrm{End}_\mathfrak{g}(A/J_{\mu,\lambda}) \simeq (A/J_{\mu,\lambda})^\mathfrak{g} \simeq A^\mathfrak{g}/J^\mathfrak{g}_{\mu,\lambda},$$

where the algebra structure is by composition in the first expression and induced by multiplication in A and $A^\mathfrak{g}$ in the last two expressions. In particular, we see that $(a_1 + J_{\mu,\lambda}) \cdot (a_2 + J_{\mu,\lambda}) = a_1a_2 + J_{\mu,\lambda}$.

Remark. We make several comments on this definition.

- The first two expressions for $R(\mathfrak{g}, A, \lambda)$ may be defined for any Lie algebra $\mathfrak{g}$ and are isomorphic without any assumption on $\mathfrak{g}$ and μ.
- The second isomorphism sends $\phi \in \mathrm{End}_\mathfrak{g}(A/J_{\mu,\lambda})$ to $\phi(1) \in (A/J_{\mu,\lambda})^\mathfrak{g}$, where we note that $\phi(1)$ is $\mathfrak{g}$-invariant because 1 is.
- In general, $J_{\mu,\lambda}$ is not a two-sided ideal in A, so $A/J_{\mu,\lambda}$ does not carry an algebra structure.
- The final isomorphism relies on the fact that $\mathfrak{g}$ is a reductive Lie algebra acting locally finitely and completely reducibly on A. In this case, Lemma 2.1 shows that $J_{\mu,\lambda}$ becomes a two-sided ideal after taking $\mathfrak{g}$-invariants, so we may consider the final expression as a quotient algebra of $A^\mathfrak{g}$.

Lemma 2.1. The invariant ideal $J^\mathfrak{g}_{\mu,\lambda}$ is a two-sided ideal in $A^\mathfrak{g}$.

Proof. For any $b \in A^\mathfrak{g}$, we have $[\mu(x) - \langle\lambda, x\rangle, b] = 0$ by definition, hence for any element $\xi = a(\mu(x) - \langle\lambda, x\rangle) \in J^\mathfrak{g}_{\mu,\lambda}$, we see that $\xi b = ab(\mu(x) - \langle\lambda, x\rangle) \in J^\mathfrak{g}_{\mu,\lambda}$. □

If $A_\hbar$ is a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}[\hbar]$-algebra, we may define the homogenized reduction $R_\hbar(\mathfrak{g}, A_\hbar, \lambda)$ in the same way by replacing $J_{\mu,\lambda}$ by $J_{\mu,\lambda,\hbar}$ in the discussion above. Note that $R_\hbar(\mathfrak{g}, A_\hbar, \lambda)$ is $\mathbb{Z}_{\geq 0}$-graded because the $\mathfrak{g}$-action preserves grading and $J^\mathfrak{g}_{\mu_\hbar,\lambda,\hbar}$ is a graded ideal.

2.3. **Universal reduction.** For the rest of the talk, it will be useful to consider the *universal reduction* of A with respect to μ, which is the algebra defined by

$$R(\mathfrak{g}, A) := A^{\mathfrak{g}}/(A\mu([\mathfrak{g},\mathfrak{g}]))^{\mathfrak{g}}.$$

Multiplying on the left via $\mu : U(\mathfrak{g}) \to A$ and projecting to $\mathfrak{g}$-invariants gives a $U(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$-action on $R(\mathfrak{g}, A)$, where we note that $U(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]) \simeq S(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$. Further, we see that

$$R(\mathfrak{g}, A, \lambda) = \mathbb{C}_\lambda \underset{S(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])}{\otimes} R(\mathfrak{g}, A),$$

where $\mathbb{C}_\lambda$ is the $S(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$-module associated with the character λ. In a similar way, we may define $R_\hbar(\mathfrak{g}, A_\hbar)$, which is a $S_\hbar(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$-algebra so that

$$R_\hbar(\mathfrak{g}, A_\hbar, \lambda) = \mathbb{C}_\lambda[\hbar] \underset{S_\hbar(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])}{\otimes} R_\hbar(\mathfrak{g}, A_\hbar).$$

3. Quantization commutes with reduction

3.1. **Classical reduction.** We now wish to view the quantum reduction as a quantization of the classical Hamiltonian reduction. Let (M, ω) be a symplectic algebraic variety with G-action which induces a map of Lie algebras $\phi_0 : \mathfrak{g} \to \mathrm{Der}(\mathbb{C}[M])$. A *classical co-moment map* μ_0 may be viewed as a map of Poisson algebras

$$\mu_0 : S(\mathfrak{g}) \to \mathbb{C}[M]$$

so that $\{\mu_0(x), f\} = \phi_0(x) \cdot f$. Define the left Poisson ideal

$$I_{\mu_0} := \mathbb{C}[M] \cdot \mu_0(\mathfrak{g})$$

Recall that the *classical Hamiltonian reduction* of M with respect to λ is

$$R(G, M, 0) := (\mathbb{C}[M]/I_{\mu_0})^G \simeq \mathbb{C}[M]^G/I_{\mu_0}^G.$$

Geometrically, it corresponds to the space $(\mu_0^*)^{-1}(0)//G = \mathrm{Spec}(R(G, M, 0))$.

3.2. **Quantum reduction as a quantization.** We now relate the quantum and classical reductions. We will consider mainly the graded context, but will provide the corresponding statements in the filtered context. Let $A_\hbar$ be a graded quantization of $A_0 := \mathbb{C}[M]$. If $\mu_\hbar : U_\hbar(\mathfrak{g}) \to A_\hbar$ is a homogenized quantum moment map, then $\mu_0 := \mu \pmod{\hbar}$ is a classical co-moment map for the G-action on M. Under certain conditions, Proposition 3.1 allows us to relate the two types of reduction.

Proposition 3.1. Let $I_0 \subset A_0$ be the ideal generated by $\mu_0(\mathfrak{g})$. Let $x_1, \ldots, x_k$ be a basis for $\mathfrak{g}$. If $\mu_0(x_1), \ldots, \mu_0(x_k)$ form a regular sequence in A_0, then for any λ, the quantum reduction $R_\hbar(\mathfrak{g}, A_\hbar, \lambda)$ is a graded quantization of the classical reduction $R(G, M, 0)$.

Proof. Write $\mathfrak{z} = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ and consider the decomposition $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus \mathfrak{z}$. Note that $\mu_0(x_1), \ldots, \mu_0(x_k)$ is a regular sequence if and only if $\mathrm{Spec}(A_0/I_0)$ has codimension k in $\mathrm{Spec}(A_0)$, so we may assume the basis $\{x_i\}$ is chosen so that $x_1, \ldots, x_m$ form a basis of $[\mathfrak{g},\mathfrak{g}]$ and $x_{m+1}, \ldots, x_k$ form a basis of $\mathfrak{z}$.

We first claim that $I = A_\hbar \mu([\mathfrak{g},\mathfrak{g}])$ is saturated, meaning that $I \cap \hbar A_\hbar = \hbar I$. Suppose that $\hbar a \in I$ for some $a \in A_\hbar$; then we can write $\hbar a = \sum_i a_i \mu(x_i)$ for some $a_i \in A_\hbar$. Passing to the quotient, we see that $\sum_i a_i^0 \mu_0(x_i) = 0$, where a_i^0 denotes the image of a_i in A_0. Because $\mu_0(x_i)$ form a regular sequence, the first homology of the associated Koszul complex

$$\bigoplus_{i<j} A_0 \overset{\sum_{i\neq j} a_{ij}^0 \mu_0(x_j)}{\to} \bigoplus_i A_0 \overset{\sum_i a_i^0 \cdot \mu_0(x_i)}{\to} A_0$$

is trivial, where we adopt the convention that $a_{ji}^0 = -a_{ij}^0$ for $i < j$. Therefore, we may write

$$a_i^0 = \sum_{i \neq j} a_{ij}^0 \mu_0(x_j)$$

for some a_{ij}^0. We conclude that $a_i = \sum_{i\neq j} a_{ij}\mu(x_j) + \hbar b_i$ for lifts a_{ij} of a_{ij}^0 and $b_i \in A_\hbar$, hence

$$a = \sum_i \sum_{j \neq i} a_{ij}\mu(x_j)\mu(x_i) + \hbar \sum_i b_i \mu(x_i) = \sum_{i<j} a_{ij}[\mu(x_j), \mu(x_i)] + \hbar \sum_i b_i \mu(x_i) \in \hbar I,$$

since $[\mu(x_i), \mu(x_j)] = \hbar\mu([x_i, x_j])$. We obtain the desired $I \cap \hbar A_\hbar = \hbar I$.

We claim now that $A_\hbar/I$ is graded free over $S_\hbar(\mathfrak{z})$. Saturation and the given regularity condition mean exactly that $\hbar, \mu_0(x_{m+1}), \ldots, \mu_0(x_k)$ form a $A_\hbar/I$-regular sequence, so this follows from Lemma 3.2. We conclude further that $R_\hbar(\mathfrak{g}, A) = (A_\hbar/I)^{\mathfrak{g}}$ is graded free over $S_\hbar(\mathfrak{z})$ as well because the $\mathfrak{g}$-action commutes with the action of $S_\hbar(\mathfrak{z})$.

Lemma 3.2. Let M be a graded $k[x_1, \ldots, x_n]$-module so that $x_1, \ldots, x_n$ are a M-regular sequence and $\deg(x_i) = 1$. Then M is a (graded) free module over $k[x_1, \ldots, x_n]$.

Proof. By induction on n, we may reduce to the case where $n = 1$, where the statement is obvious. □

Observe now that $R_\hbar(\mathfrak{g}, A, \lambda) = \mathbb{C}_\lambda[\hbar] \underset{S_\hbar(\mathfrak{z})}{\otimes} R_\hbar(\mathfrak{g}, A)$ is evidently flat over $\mathbb{C}[\hbar]$ and that

$$\begin{aligned} R_\hbar(\mathfrak{g}, A, \lambda)/\hbar R_\hbar(\mathfrak{g}, A, \lambda) &= \mathbb{C} \underset{\mathbb{C}[\hbar]}{\otimes} \mathbb{C}_\lambda[\hbar] \underset{S_\hbar(\mathfrak{z})}{\otimes} R_\hbar(\mathfrak{g}, A) \\ &= \mathbb{C} \underset{S(\mathfrak{z})}{\otimes} S(\mathfrak{z}) \underset{S_\hbar(\mathfrak{z})}{\otimes} R_\hbar(\mathfrak{g}, A) \\ &= \mathbb{C} \underset{R}{\otimes} (\mathfrak{g}, A_0) \\ &= R(G, M, 0). \end{aligned}$$

We conclude that $R_\hbar(\mathfrak{g}, A, \lambda)$ is a graded quantization of $R(G, M, 0)$ for all λ. □

Consider now the corresponding filtered setting. Let A be a filtered quantization of A_0 and $\mu : U(\mathfrak{g}) \to A$ a quantum moment map so that $\mu_0 := \mathrm{gr}(\mu)$ is a classical co-moment map. Translating Proposition 3.1 into this filtered setting, we obtain Corollary 3.3.

Corollary 3.3. Let $I_0 \subset A_0$ be the ideal generated by $\mu_0(\mathfrak{g})$. Let $x_1, \ldots, x_k$ be a basis for $\mathfrak{g}$. If $\mu_0(x_1), \ldots, \mu_0(x_k)$ form a regular sequence in A_0, then for any λ, the quantum reduction $R(\mathfrak{g}, A, \lambda)$ is a filtered quantization of the classical reduction $R(G, M, 0)$.

Proof. Each of the constructions in the filtered case is obtained by reducing the corresponding construction in the graded case modulo the ideal generated by $(\hbar - 1)$, so this follows from Proposition 3.1. □

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