SYMPLECTIC REFLECTION ALGEBRAS

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Symplectic reflection algebras are certain associative algebras introduced by P.E. and Ginzburg in the beginning of 2000's. They are connected to other objects in Representation theory (quivers, Hecke algebras) but also to Algebraic Geometry (resolutions and deformations of symplectic quotient singularities, plane curves), Combinatorics (Macdonald polynomials, Catalan numbers) and Integrable systems (Calogero-Moser type systems). In this lectures we will explain some foundations of Symplectic reflection algebras and then proceed to two topics of current research that connect Symplectic reflection algebras to affine Lie algebras in very different ways:

(1) The proofs of P.E.'s conjecture on counting representations of the cyclotomic Rational Cherednik algebras with given support (by Shan and Vasserot) and of a P.E. type conjecture on counting finite dimensional irreducible representations for quantized quiver varieties (R.B. and I.L.).
(2) The proof of a Varagnolo-Vasserot category equivalence conjecture (by I.L. and also by Rouquier, Shan, Varagnolo, Vasserot)

Besides, we will discuss some other popular and important topics in Representation theory, such as categorical Kac-Moody actions and fusion products of representations of affine Kac-Moody algebras.

We will concentrate on (1) and discuss (2) time permitting. Also there will be lectures by R.B. on a connection of P.E.'s conjectures to geometry.

Students are especially encouraged to attend!

Still preliminary (optimistic!) program:

1) Symplectic reflection algebras as deformations of skew-group algebras (P.E.).

1.1: Smash-products= skew-group algebras.
1.2: Definition of SRA.
1.3: Computation of Hochschild cohomology – basics on Hochschild cohomology is a prerequisite.
1.4: Consequence: flatness and universality of SRA.
1.5: Spherical subalgebras. Induced Poisson brackets.

2) Nakajima quiver varieties (I.L.).

2.1. Quivers and their representations.
2.2. Nakajima quiver varieties as GIT quotients (basics on GIT quotients is a prerequisite). Poisson structures.
2.3. Examples (cotangent bundles and symplectic resolutions).
2.4. Nakajima's construction of integrable representations (convolutions in cohomology – black box).
2.5. Symplectic resolutions and Maffei's isomorphisms.

3) Rational Cherednik algebras and their categories $\mathcal{O}$ (P.E.).

3.1. Rational Cherednik algebras. Triangular decomposition and the Euler element.
3.2. Dunkl operators. Localization.
3.3. Categories O. Verma and simple modules. Example: cyclotomic category $\mathcal{O}$.
3.4. Finite length. Projectives.
3.5. Highest weight structure.

3.6. Duality and costandard objects.

4) Quantizations of quiver varieties. Relation to Symplectic reflection algebras (I.L.).

4.1. Quantizations of graded algebras and quantum Hamiltonian reduction.
4.2. Isomorphism of quantum Hamiltonian reduction and SRA.
4.3. Microlocal quantizations and quantum Hamiltonian reduction.
4.4. Quantization of Maffei's construction.
4.5. Supports.
4.6. Localization theorems.
4.7. Translation functors.

5) The main results (Varagnolo-Vasserot conjecture and Etingof type conjecture). (P.E. & I.L.)

5.1. The Varagnolo-Vasserot conjecture on cyclotomic categories $\mathcal{O}$ (I.L.).
5.2. Etingof's conjecture on the number of irreducibles with given support in the cyclotomic case. (P.E.).
5.3. Etingof type conjecture for quantized quiver varieties (I.L).

6) Induction and restriction functors (P.E.).

6.1. Completions and their isomorphisms.
6.2. Construction of induction and restriction functors.
6.3. Exactness.
6.4. Behavior on K_0.
6.5. Induction and restriction vs supports.
6.6. Inductions and restrictions vs duality. Biadjointness. Behavior on standard and costandard objects.

7) Procesi bundles and their deformations (I.L.).

7.1. Definition.
7.2. Overview of an approach due to Bezrukavnikov and Kaledin.
7.3. Deformations of Procesi bundle and their endomorphisms.
7.4. Proof of an isomorphism between SRA and quantum Hamiltonian reduction.
7.5. Deformed McKay equivalences.
7.6. Classification of Procesi bundles and Macdonald positivity.

8) KZ functor and its properties (P.E.).

8.1. Construction of functor from $\mathcal{O}$ to modules over the braid group.
8.2. The action on the image factors through the Hecke algebra (via reduction to codimension 1).
8.3. The projective P_{KZ}.
8.4. The double centralizer property.
8.5. Surjectivity.
8.6. 0-faithfulness.
8.7. Relation with induction and restriction functors.
8.8. Cyclotomic Hecke algebras.

9) Rouquier's equivalences of categories (I.L., P.E.).

9.1. Classical Schur-Weyl duality (I.L.).
9.2. Quantum Schur-Weyl duality (P.E.).
9.3. Description of the images of projectives (I.L.).

10) Categorical Kac-Moody actions (I.L.).

10.1. Motivation – cyclotomic Hecke algebras.
10.2. Formal definition.
10.3. Shan's categorical action on cyclotomic categories O.
10.4. Categorified Cartan component and minimal categorifications.
10.5. Rickard complexes.
10.6. Filtration ($\mathfrak{sl}_2$-case).
10.7. Crystals.
10.8. Graded lift ($\mathfrak{sl}_2$-case).

11) Fusion products for affine Lie algebras. (P.E.)