Braid group actions on categories of coherent sheaves

MIT-Northeastern Rep Theory Seminar

In this talk we will construct, following the recent paper [BR] by Bezrukavnikov and Riche, actions of certain groups on certain categories. Both the groups and the categories are relevant in representation theory, but all currently known constructions of the action are given by explicitly constructing action by generators and checking relations between them — this is what we will do. There are conjectures giving more "direct" meanings of these actions in terms of mirror symmetry, but these remain largely mysterious at present.

Definition 1. For a category $\mathcal{C}$, write $\mathrm{AutEquiv}(\mathcal{C})$ for the category of "autoequivalences", i.e. invertible functors, from $\mathcal{C}$ to itself. Write

$$\mathrm{Ob}\left(\mathrm{AutEquiv}(\mathcal{C})\right)$$

for the set of autoequivalences considered up to automorphism (of functors). This is a group.

Definition 2. An action of a group G on a category $\mathcal{C}$ is a map of groups $\alpha : G \to \mathrm{AutEquiv}(\mathcal{C})$. Note that this is a *weak* action.

The categories we will be dealing with will be triangulated, and the functors between them will be triangulated functors. Associated to functor $\alpha(g)$ will be an endomorphism of the Grothendieck group $K^0(\mathcal{C})$.

For each group action, we will study the span of its image H in $\mathrm{End}(K^0\mathcal{C})$, and on endomorphisms of smaller invariant subcategories. These images will be important algebras in their own right, and the "canonical" representations as endomorphisms of a Grothendieck group will be important in later talks.

Thus we will produce quadruples of the form "category $\mathcal{C}$, group G, algebra H, representations V_e" (classified by nilpotent orbits).

We will go through four "variations" of this construction, giving the talk a flavor of the musical movement called "theme and variations". The theme, which is self-contained, will contain the essential construction that will later be elaborated in the variations. In the first setting of the "theme" the group G will be the braid group, the category — that of (derived) coherent sheaves on $\tilde{N}$, possibly with support conditions, and the algebra will be the group algebra of the Weyl group, $\mathbb{C}[W]$.

1 Theme: A classical braid group action on $\mathrm{coh}(\tilde{N})$.

Fix a root system corresponding to a finite-dimensional Lie algebra $\mathfrak{g}$

Definition 3. The *braid group* Br of a root system is defined as $\pi_1(\big(\mathfrak{h}\backslash\mathfrak{h}_\alpha\big)/W)$, where $\mathfrak{h}$ is the Cartan Lie subalgebra, W is the Weyl group and $\mathfrak{h}_\alpha$ are the root hyperplanes corresponding to all roots.

We will be using several different presentations of the braid group. First, we recall a presentation for the Weyl group W: Take for generators the w_α corresponding to simple reflections α, for α running over simple roots.

Lemma 4. *The w_α generate the Weyl group W. For g, h group elements, define the word $\{g, h\}^m = ghghg\ldots$, where there are m letters (e.g. if m is odd, the last letter is an a). For a pair of roots α, β, let $m(\alpha, \beta)$ be the denominator of the angle $\frac{\angle(\alpha,\beta)}{\pi}$. The following is a complete set of relations on the w_α:*

$$\forall \alpha, \beta : \tag{1}$$

$$w_\alpha^2 = 1 \tag{2}$$

$$\{w_\alpha, w_\beta\}^{m(\alpha,\beta)} = \{w_\beta, w_\alpha\}^{m(\alpha,\beta)}. \tag{3}$$

Now we can get a presentation of the braid group, by just getting rid of one of the relations.

Lemma 5. *The braid group* Br *can be presented with generators w_α, where α runs over* simple *roots and*

$$\forall \alpha, \beta : \tag{4}$$

$$\{T_\alpha, T_\beta\}^{m(\alpha,\beta)} = \{T_\beta, T_\alpha\}^{m(\alpha,\beta)}. \tag{5}$$

Here the generators T_α correspond to a half-turn around one of the hyperplanes in $\check{\mathfrak{h}}$. Note that if α, β are orthogonal then $m(\alpha, \beta) = 2$ and the relation says T_α, T_β commute. For type A_n the roots are indexed by $1, \ldots, n$ and relations are $T_iT_j = T_jT_i$ for $i \neq j \pm 1$ and $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, the usual relation for the ordinary braid group.

We see from the presentations that Br surjects onto W. This map comes from the topological fact that $\pi_1(X/W)$ maps to W for any free action of a group W on a topological space X (this map is surjective so long as X is connected).

One more presentation of the Braid group will be relevant below, where instead of taking genrerators corresponding to simple reflections we take generators corresponding to certain lifts of *all* elements of the Weyl group W. Recall the notion of *length* of an element $w \in W$, equal to the length of the minimal length of a presentation of w as a product of simple reflections.

Lemma 6. *Let T_w be a set of generators indexed by $w \in W$. Then* Br *has a presentation as follows.*

$$T_{w_1}T_{w_2} = T_{w_1w_2} \text{ as long as } l(w_1) + l(w_2) = l(w_1w_2). \tag{6}$$

To see this is equivalent to 5, write T_w as $\prod_i T_{\alpha_i}$ for $\alpha_1\alpha_2\ldots$ a shortest presentation of w as a product of simple reflections. Equivalence follows from the fact that any shortest presentation for w can be obtained from any other by using the braid relation (3) from Lemma 4.

1.1 Affine analogues

Though we won't need it till the next section, it is appropriate here we make a remark collecting analogues of the facts above for affine coxeter systems:

Proposition-Definition 7. *1. Let Δ be a set of simple roots of a coxeter system for an affine Dynkin diagram of the group $\hat{\mathfrak{g}}$ ($\mathfrak{g}$ classical). Then $w_\alpha, \alpha \in \Delta$ with the relations from Lemma 4 span the* Affine Weyl group.

2. *The w_α corresponding to the classical vertices of the Dynkin diagram span a subgroup isomorphic to W and the affine Weyl group W_{Aff} of $\hat{\mathfrak{g}}$ is isomorhpic to $\Lambda \ltimes W$ where Λ is the weight system of the (classical) Weyl group for $\mathfrak{g}$.*

3. *(A presentation of Br_{Aff}). The generators and relations in 5 above, applied to the coxeter system for $\hat{\mathfrak{g}}$ span the* Affine Braid group, *Br_{Aff}.*

4. *There is an alternative presentation for Br_{Aff} analogous to Lemma 6, with generators T_w for $w \in W \subset W_{Aff}$ and v_λ for $\lambda \in \Lambda \subset W_{Aff}$, with the following relations.*

$$T_\alpha v_{s_\alpha(\lambda)} T_\alpha = v_\lambda, \text{ when } (\lambda, \check{\alpha}) = 1 \tag{7}$$
$$T_\alpha v_\lambda = T_\alpha v_\lambda, \text{ when } (\lambda, \check{\alpha}) = 0 \tag{8}$$

in addition to the old relation for Br

$$T_{w_1} T_{w_2} = T_{w_1 w_2}, \text{ when } l(w_1) + l(w_2) = l(w_1 w_2) \tag{9}$$

and the usualy multiplication relations for the lattice $\Lambda \subset Br_{Aff}$, namely

$$v_\lambda v_{\lambda'} = v_{\lambda+\lambda'}. \tag{10}$$

This is written in the appendix of [BR].

1.2 Grothendieck groups

For $\mathcal{C}$ an abelian category, let $K^0(\mathcal{C})$, the K-group, or Grothendieck group of $\mathcal{C}$ denote the quotient of the set $\{[A] \mid A \in \mathrm{Ob}(\mathcal{C})\}$ by the relation of "additivity": $[A] + [C] = [B]$ for $0 \to A \to B \to C \to 0$ exact.

Remark 8. The superscript 0 comes from the fact that there are higher K groups, and these can be useful for constructing long exact sequences. We will not encounter them here.

Similarly, for $\mathcal{D}$ a triangulated category, we define $K^0(\mathcal{D})$ the category generated by $[A]$ modulo the relation $[A]+[C]=[B]$ for $A \to B \to C \to$ an exact triangle.

We have the following simple lemma:

Lemma 9. $K^0(\mathcal{C}) = K^0(D^b\mathcal{C})$.

When X is a variety, we can define two derived categories of coherent sheaves on X, "bounded" in different ways. One is $D^b \operatorname{coh}(X)$, the derived category of the category of coherent sheaves on X. The other is $D^b \operatorname{bun}(X)$, the full subcategory of $D^b \operatorname{coh}(X)$ spanned by locally free sheaves. Hence we define two groups as follows.

Definition 10. $K_0(X) = K^0(\operatorname{coh}(X))$ and
$K^0(X) = K^0(\operatorname{bun}(X))$.

The subscript in K_0 may seem strange, but comes from the analogy between the pair $(K_0(X), K^0(X))$ and the pair $H_*(X), H^*(X)$ (see Proposition-Definition 11). When X is smooth, we have

$$K^0(X) \cong K_0(X)$$

since any coherent sheaf has a finite locally free resolution.

Proposition-Definition 11. *1. For $f : X \to Y$ a proper map, pushforward $f_* : D^b \operatorname{coh}(X) \to D^b \operatorname{coh}(Y)$ induces by linearity a map $f_* : K_0(X) \to K_0(Y)$*

2. and for $f : X \to Y$ any map, $f^* : D^b \operatorname{bun}(Y) \to D^b \operatorname{bun}(X)$ induces a map of rings $f^* : K^0(X) \to K^0(X)$.

3. The assignment $[V]\cdot[W] = [V \otimes W]$ for V a locally free bundle defines (by linearity) a ring structure $\cdot : K^0(X) \otimes K^0(X) \to K^0(X)$ and an action $\cdot : K^0(X) \otimes K_0(X) \to K_0(X)$.

In terms of this analogy, the identification $K^0(X) \cong K_0(X)$ for X smooth corresponds to Poincaré duality.

For X an algebraic variety there is a map called the (topological) Chern character, $\chi : K^0(X) \to \prod_i H^i(X)$ (valued in formal series in cohomology groups). We define $\chi(V) = \chi(V_{\text{top}})$, where for V a locally free sheaf, V_{top} is the topological bundle on the topological space X corresponding to V.

Now we have *Chern character* maps, $\chi^0 : K^0(X) \to H^*(X, \mathbb{Q})$ ("cohomological"), and $\chi_0 : K_0(X) \to \sum_i H_i^{BM}(X, \mathbb{Q})$ ("homological"). The maps χ_0, χ^0 satisfy the following properties:

1. $\chi^0 : K^0(X) \to H^*(X)$ is a map of rings
2. $\chi_0 : K_0(X) \to H_*(X)$ is a map of modules
3. χ^0 commutes with pullbacks.

4. the Grothendieck-Riemann-Roch theorem, a.k.a. “twisted” compatibility with pushforward:

$$\mathrm{Td}_Y \cdot f_*\chi_0(F) = f_*(\mathrm{Td}_X \cdot \chi_0(F)).$$

For the last equation, the *Todd class* $\mathrm{Td}_X = 1 + O(c_i(T_X))$ of a smooth variety X is defined to be a certain linear combination of the Chern classes of its tangent bundle.

We will need a final fact: χ_0 is compatible with the support filtration on K_0, where for an integer d we say that $\sum m_i[F_i] \in \tau_{\leq d}K^0$ whenever the sheaves F_i have support of dimensions $d_i \leq d$.

In this case, we have the following lemma.

Lemma 12. *χ_0 is compatible with the filtration $\tau_{\leq d}(H^*(X)) := \sum_{i\leq d} H^i(X)$. Further, the map $gr(\chi_0) : gr(K_0(X), \tau_{\leq *}) \to H_*(X)$ commutes with pushforwards.*

Here the Todd class factor Td_X can be ignored as it acts by 1 on $H_*(X)$ after taking associated gradeds.

1.3 Statements of results for classical braid groups.

We can now state the main result of the paper [BR] for the classical braid group:

Theorem 13. *There is a natural action of the braid group* Br *on the category $D^b\,\mathrm{coh}(\tilde{N})$, taking the generators T_w to functors Γ_w (to be defined in the next section).*

Further, this action is in fibered over the nilpotent cone N in the following sense. Namely, for any nilpotent $e \in N$, define $\mathbb{B}_e \subset \tilde{N}$ to be the Springer fiber over e. Then the action by Br *preserves the derived category $D^b\,\mathrm{coh}_{\mathbb{B}_e}(\tilde{N})$ of coherent sheaves with cohomology supported on $\mathbb{B}_e$.*

Finally, the action on the level of the Grothendieck group $K_0(\mathbb{B}_e)$ factors through the projection $\mathrm{Br} \to W$.

In particular, we get an action of W on $K^0(D^b\,\mathrm{coh}_{\mathbb{B}_e}(\tilde{N})) = K_0(\mathbb{B}_e)$. We have

$$K_0(\mathbb{B}_e) \otimes \mathbb{Q} \cong H_*(\mathbb{B}_e)$$

(this is in [CG]).

We can ask whether the Chern character $\chi_0 : K_0(\mathbb{B}_e) \to H_*^{BM}(\mathbb{B}_e)$ intertwines the action of W defined above with the action defined in [CG] that Yi talked about in the last talk. It turns out that this is not the case[1] However we have the following.

Theorem 14. *The Weyl group action is compatible with the support filtration $\tau_{\leq *}K_0(\mathbb{B}_e)$, and on the level of associated gradeds $\mathrm{gr}\,\chi_0 : \mathrm{gr}\,K_0(\mathbb{B}_e) \to H_*^{BM}(\mathbb{B}_e)$ is compatible with the W action on H_*^{BM} defined in the last lecture.*

[1] In the talk on 9/16 I made the opposite claim. Thanks to Paul Seidel for catching the mistake.

1.4 Constructing the representation.

There is a standard way of constructing triangulated functors $D^b\,\mathrm{coh}(X) \to D^b\,\mathrm{coh}(Y)$: pick a coherent sheaf $K \in \mathrm{coh}(X \times Y)$, such that it has a finite resolution by locally free sheaves.

Definition 15. Define the *Fourier-Mukai* transformation with *Kernel* K, denoted $FM(K)$, to be the composition functor

$$FM : d^b\,\mathrm{coh}(X) \xrightarrow{\pi_1^*} D^b\,\mathrm{coh}(X \times Y) \xrightarrow{\otimes K} D^b\,\mathrm{coh}(X \times Y) \xrightarrow{R(\pi_2)_*} \mathrm{coh}(Y),$$

with $F \mapsto (\pi_2)_*(K \otimes \pi_1^* F)$.

Remark 16. The "Fourier" in "Fourier-Mukai" comes from the following analogy: if we replaced "pushforward" by "integral" and the Kernel K by $e^{x\cdot y}$, we would have the ordinary Fourier transform. Note that composition corresponds to convolution, $FM(K)\circ FM(K') = FM(K * K')$, which, again in analogy with integrals, corresponds to multiplication of matrices (with entries parametrized by $X \times Y$).

Definition 17. 1. If $\iota : Z \subset X \times Y$ is a closed subset, define $FM(Z) = FM(\iota_* O(Z))$.

2. If M is a sheaf over X, define $FM(M) = \Delta_*(M)$. Evidently,

$$FM(M) : N \mapsto M \overset{L}{\otimes} N$$

for M another sheaf.

Note that if $\Gamma \subset X \times Y$ is a graph of a map $X \to Y$ then $FM(\Gamma) = f^*$.

We define the functor $\Phi_w : D^b\,\mathrm{coh}(\tilde{N}) \to D^b\,\mathrm{coh}(\tilde{N})$ by $\Phi_w = FM(\Lambda_0^w)$, where Λ_0^w were the spaces defined in the previous talk (denoted Z_w in [BR]).

Theorem 18 (Bezrukavnikov-Riche)**.** *1. The Φ_w are invertible;*

2. We have a canonical isomorphism of functors

$$\Phi_{w_1}\Phi_{w_2} \cong \Phi_{w_1 w_2} \text{ if } l(w_1) + l(w_2) = l(w_1 w_2).$$

The fact that $l(w_1)l(w_2) = l(w_1 w_2)$ is necessary can be seen by projecting $\tilde{N} \to \mathbb{B}$. Then the set-theoretic images of Λ_0^w are the Bruhat double coset O_w, and on the level of sets, we have $O_{w_1} * O_{w_2} = O_{w_1 w_2}$ iff $l(w_1 w_2) = l(w_1) + l(w_2)$. The proof that this is sufficient is much more mysterious, and proceeds by reduction to characteristic p.

1.5 Action on K_0 and standard modules

Finally, recall from the previous talk that $\Lambda_0^w \subset \tilde{N} \times \tilde{N}$ for $w \in W$ is the fiber over $0 \in \mathfrak{h} \times \mathfrak{h}$ of a space $\Lambda^w \subset \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$ (where the map $\tilde{\mathfrak{g}} \to \tilde{\mathfrak{h}}$ is the Grothendieck simultaneous resolution). Now over the regular subset $\mathfrak{h}^{\text{reg}} \times \mathfrak{h}^{\text{reg}} \subset \mathfrak{h} \times \mathfrak{h}$ (which is dense), the preimage $\Lambda^w|_{\mathfrak{h}^{\text{reg}}} \times \mathfrak{h}^{\text{reg}}$ is the graph of a natural action of $w \in W$ on $\tilde{\mathfrak{g}}_{\mathfrak{h}^{\text{reg}}}$. In particular, on this open the functors $\Phi_w^{\text{reg}} := FM\big(\Lambda^w|_{\mathfrak{h}^{\text{reg}}} \times \mathfrak{h}^{\text{reg}}\big)$ satisfy $\Phi_{w_1} \circ \Phi_{w_2} = \Phi_{w_1 w_2}$. Now K_0 admits a specialization map like the one Yi defined in the last talk on H_*^{BM}, and so from the isomorphism of functors $\Phi_{w_1}^{\text{reg}} \Phi_{w_2}^{\text{reg}} \cong \Phi_{w_1 w_2}^{\text{reg}}$ (or rather from equality of the maps they induce on K_0) we can deduce that $K_0(\Phi_{w_1}\Phi_{w_2}) = K_0(\Phi_{w_1 w_2})$ without any conditions on length, and so the induced action of Br on $K_0(\tilde{N})$ factors through W.

An analogous argument can be used to show that the action on the category of coherent sheaves with support restriction, $K^0(\mathrm{coh}_{\mathbb{B}_e}(\tilde{N})$, factors through W. We have an identification

$$K^0(D^b \,\mathrm{coh}_{\mathbb{B}_e}(\tilde{N})) \cong K^0(D^b \,\mathrm{coh}\, \mathbb{B}_e),$$

as any module supported on $\mathbb{B}_e$ has a filtration by finitely many pushforwards of modules from $\mathbb{B}_e$. Hence W acts on $K^0 D^b \,\mathrm{coh}(\mathbb{B}_e) = K_0(\mathbb{B}_e)$. We define the *standard modules* of W corresponding to Springer fibers $\mathbb{B}_e$ to be the $K_0(\mathbb{B}_e) \otimes \mathbb{Q}$ with W-action induced by the Φ_w.

2 Variations: Equivariant coherent sheaves: G-equivariance

We're done with our theme. Now on to variations. These will require a good understanding of the category of equivariant modules over a scheme with a $\mathcal{G}$ action for an algebraic group $\mathcal{G}$.

Suppose R is a commutative ring with action by an algebraic group $\mathcal{G}$ (so denoted to distinguish it from our original Lie group G). Let $\alpha(g) : R \to R$ (map of rings) be this map. Say M is a module over R.

Definition 19. A $\mathcal{G}$-equivariant structure on M is a $\mathcal{G}$-action on M such that

$$(gr)(gm) = g(rm) \tag{11}$$

For an automorphism $\alpha(g)$ of R, the pushforward $\alpha(g)_*(M)$ twists the action of R by M by $\alpha(g) : R \to R$. Formula (11) is equivalent to the g-action on M defining a map $M \to \rho(g)_*(M)$. Now suppose X is a scheme with action by a group $\mathcal{G}$. We modify the above definition of an equivariant module as follows.

Definition 20. Given a scheme X with action $\alpha : \mathcal{G} \to \mathrm{Aut}(X)$, an $\mathcal{G}$-*equivariang sheaf over* X is a coherent sheaf M together with a collection of isomorphisms $\rho(g) : M \to \alpha(g)_* M$ with the following compatibility condition:

$$\big(\alpha(g_1)_*[\rho(g_2)]\big)\rho(g_1) = \rho(g_2 g_1) : M \to \alpha(g_2 g_2)_*(M). \tag{12}$$

It goes without saying that both the action $\alpha(g)$ on X and the mapping $g \mapsto \rho(g)$ must be algebraic in g in the obvious sense.

The category of $\mathcal{G}$-equivariant sheaves as above is abelian and monoidal and will be denoted $\mathrm{coh}^{\mathcal{G}}(X)$. When X is affine and M is a module, the above simply encodes the associativity condition on the action ρ of $\mathcal{G}$ on M.

Remark 21.

In the following we will give several ways of constructing classes of equivariant sheaves, which we will use for our construction of new functors from $\mathrm{coh}^{\mathcal{G}}(\tilde{N})$ to itself.

Definition 22. Suppose V is a representation of $\mathcal{G}$. Consider the sheaf $O(V) = O_X \otimes V$. Then for any map $f : X \to X$ — in particular, for action maps $f = \alpha(g^{-1})_*$, we have canonically $f^*O(V) \cong O(V)$. Using this identification, define the equivariance maps $\rho(g) : O(V) \to O(V)$ using the representation action of g on V.

As a special case of this when $\mathcal{G} = \mathbb{C}^*$, the character sheaf $O(\chi)$ given by the one-dimensional representation $\chi : \mathbb{C}^* \to \mathbb{C}$.

Another case of interest for us is the character $\chi_\lambda : B \to \mathbb{C}^*$ given by pulling back to B a weight $\lambda : T \to \mathbb{C}^*$. Namely, note that G-equivariant line bundles on G/B are in bijection with characters of B, where the bundle L is determined by its coherence maps $\rho_b : L_{[1]} \to L_{[1]}$ over the point $[1] \in G/B$ and b varies over B (the centralizer of $[1] \in G/B$).

Write $O(\lambda)$ for the bundle corresponding to $\lambda : B \to \mathbb{C}^*$. (This is a very classical definition: for example for $SL_2/B \cong \mathbb{P}^1$, these are powers of the canonical bunle, $O(n)$.) Abusing notation, we also denote by $O(\lambda)$ the bundle over $\tilde{N}$ obtained by pulling back along the map $\tilde{N} \to G/B$.

Now we define functors $\Theta_\lambda : \mathrm{coh}(\tilde{N}) \to \mathrm{coh}(\tilde{N})$ to be twists by the line bundles $O(\lambda)$. These satisfy $\Theta(\lambda) \circ \Theta(\lambda') = \Theta(\lambda\lambda')$, and in particular the functors $\Theta(\lambda)$ span a lattice isomorphic to the root lattice R.

Theorem 23 ([BR])**.** *The assignment $\mathbf{J}(T_w) = \Phi_w$ and $\mathbf{J}(T_{v_\lambda}) = \Theta_\lambda$ extends to a reprsentation of the affine braid group $\mathbf{J} : Br_{Aff} \to AutEquiv(\mathrm{coh}(\tilde{N}))$. Further,*

1. *this action respects the condition of being supported at a Springer fiber $\mathbb{B}_e$, and*

2. *For any $\mathcal{G} \subset G \times \mathbb{C}$, there are compatible representations $\mathbf{J}^{\mathcal{G}} : Br_{Aff} \to AutEquiv(\mathrm{coh}^{\mathcal{G}}(\tilde{N}))$ for $\mathcal{G}$ any subgroup of $G \times \mathbb{C}$ (we will be interested in the subgroups $G, \mathbb{C}^*, G \times \mathbb{C}^*$). Here recall that the Lie group G acts on $\tilde{N}$ by conjugation and $\mathbb{C}^*$ acts on $\tilde{N}$ by scaling, and these actions commute.*

3. *For any subgroup $\mathcal{G} \subset G \times \mathbb{C}^*$ which stabilizes a nilpotent $e \in N$, there is a canonical extension of $\mathbf{J}$ to $\mathbf{J}^{\mathcal{G}}_e : Br_{Aff} \to AutEquiv(\mathrm{coh}_{\mathbb{B}_e}(\tilde{N}))$.*

The equivariant actions above follow from the fact that both $\mathbf{J}(T_w) = FM(\Lambda_0^w)$ and $\mathbf{J}(T_{v_\lambda}) = FM(O(\lambda))$ come from the Fourier-Mukai transforms with sheaves that have a natural G-equivariant structure ($O(\lambda)$ is equivariant from the way we defined it, and Λ_0^w is a $G \times \mathbb{C}^*$-invariant subspace of $\tilde{N} \times \tilde{N}$).

Note that the braid group relations for the equivariant case do not imply them for the non-equivariant one, or vice versa — hence they need to be checked for each case separately.

Finally, we've already seen that the Φ_w preserve support conditions; the Θ_λ do as well, since tensoring with a line bundle cannot increase support.

References

[BR] Bezrukavnikov-Riche, `arXiv:1101.3702`

[CG] Representation Theory and Complex Geometry by Neil Chriss and Victor Ginzburg