Lecture 3

1) Completions of quantizations
2) Quantum slices.
3*) Finite W-algebras

1.0) Motivations.

In symplectic/Poisson C^∞-geometry we have a number of local structure results. The most basic result is the Darboux theorem: $\forall$ point m in a symplectic manifold M has a neighborhood symplectomorphic to a neighborhood of 0 in $T_m M$. There's no such description for Poisson manifolds but one has a partial result: for a point m in a Poisson manifold M we can consider a symplectic leaf $\mathcal{L}$ through m: the locus where we can get from m using Hamiltonian flow. Then a small transverse slice S to $\mathcal{L}$ at m has a Poisson structure s.t. $\exists$ neighborhoods $U \subset M$, $U_0 \subset \mathcal{L}$ of m s.t. $U \simeq U_0 \times S$ as Poisson manifolds.

We want an analog of these claims for quantizations & in the algebraic setting, where we are forced to work w. formal neighborhoods so we need to discuss completions for formal quantizations.

1) Completions of quantizations

Let A be a Poisson algebre/$\mathbb{C}$ & $\mathfrak{m}\subset A$ be a maximal ideal. Then the completion $\hat{A}=\varprojlim_n A/\mathfrak{m}^n$ has a unique Poisson bracket extended from A by continuity. Assume A is Noetherian.

Now suppose $\mathcal{A}_\hbar$ is a formal quantization of A. Let $\mathfrak{m}_\hbar$ denote the preimage of $\mathfrak{m}$ in $\mathcal{A}_\hbar$ (in particular, $\hbar\in\mathfrak{m}_\hbar$), a maximal 2-sided ideal. Then we can form the inverse limit:

$$\hat{\mathcal{A}}_\hbar=\varprojlim_n \mathcal{A}_\hbar/\mathfrak{m}_\hbar^n$$

Fact: $\hat{\mathcal{A}}_\hbar$ is a formal quantization of $\hat{A}$.

Remark on proof: The only nontrivial claim here is that $\hat{\mathcal{A}}_\hbar$ is flat over $\mathbb{C}[[\hbar]]$. This is because $\mathfrak{m}_\hbar$ satisfies the Artin-Rees lemma, which follows from more general results in Non-commutative algebra (Exer 19, §3.5, in Rowen's "Ring theory, Vol 1). $\square$

Remarks: 1) Suppose A is graded, $\mathfrak{m}$ is a homogeneous ideal & $\mathcal{A}_\hbar$ is a graded. Then $\hat{\mathcal{A}}_\hbar$ is strictly speaking not a graded formal quantization (even $\hat{A}$ is not a graded algebra). But we have compatible gradings on the quotients $\mathcal{A}_\hbar/\mathfrak{m}_\hbar^k$, cf. the definition of grading on $\mathcal{A}_\hbar$.

2) If A is finitely gen'd & $\mathfrak{m}$ corresponds to a smooth point, then $\hat{A}$ is the algebra of formal series $\mathbb{C}[[(\mathfrak{m}/\mathfrak{m}^2)^*]]$. One should view $\hat{\mathcal{A}}$ as the space $\mathbb{C}[[(\mathfrak{m}/\mathfrak{m}^2)^*]][[\hbar]]$ w. deformed product.

2) Quantum slices.

Let $A, \mathfrak{m}$ be as in the beginning of the previous section. It's convenient for us to consider a slightly different version of quantization $\mathcal{A}_\hbar$: we assume that $[\mathcal{A}_\hbar, \mathcal{A}_\hbar] \subset \hbar^2 \mathcal{A}_\hbar$.
For $a, b \in \mathcal{A}_\hbar$, set $\{a,b\} := \frac{1}{\hbar^2}[a,b]$. This is <u>not</u> a Poisson bracket on $\mathcal{A}_\hbar$ (it's not commutative) but it's a Lie bracket & $\{a,\cdot\}$ is a derivation $\forall a \in \mathcal{A}_\hbar$. We assume that $\{\cdot,\cdot\}$ on A is induced by $\{\cdot,\cdot\}$ on $\mathcal{A}_\hbar$.

Set $\tilde{V} := \mathfrak{m}/\mathfrak{m}^2$, then $\{\cdot,\cdot\}$ induces a skew-symmetric form ω on $\tilde{V}$: $\{a+\mathfrak{m}^2, b+\mathfrak{m}^2\} := \{a,b\}+\mathfrak{m}$. Pick a complement $V\subset\tilde{V}$ to the radical of ω.

Let $\hat{\mathfrak{m}}_\hbar$ & $\hat{\mathfrak{m}}$ denote the max. ideals in $\hat{\mathcal{A}}_\hbar, \hat{A}$. Consider the composition $\pi: \hat{\mathfrak{m}}_\hbar \twoheadrightarrow \hat{\mathfrak{m}} \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2 = V$

Theorem (quantum slice): $\exists\ \iota: V \hookrightarrow \mathfrak{m}_\hbar$ s.t.

1) $\pi\circ\iota = id_V$

2) $\{\iota(u),\iota(v)\} = \omega(u,v)\ \forall\ u,v\in V$. In particular, ι induces an algebra homomorphism $\hat{W}_\hbar(V)\longrightarrow\hat{\mathcal{A}}_\hbar$, where $\hat{W}_\hbar(V)$ stands for the formal Weyl algebra: the completion of the Weyl algebra $W_\hbar(V) = T(V)[\hbar]/(u\otimes v - v\otimes u - \hbar^2\omega(u,v))$ at the maximal ideal of $0\in V^*$.

3) Let $\mathcal{A}_\hbar'$ denote the centralizer of $\iota(V)$ in $\hat{\mathcal{A}}_\hbar$. Then the multiplication homomorphism $\hat{W}_\hbar(V)\otimes_{\mathbb{C}[[\hbar]]}\mathcal{A}_\hbar' \longrightarrow \hat{\mathcal{A}}_\hbar$ extends to an isomorphism $\hat{W}_\hbar(V)\hat{\otimes}_{\mathbb{C}[[\hbar]]}\mathcal{A}_\hbar' \longrightarrow \hat{\mathcal{A}}_\hbar$, where in the source we have the completed tensor product: the usual tensor product of two topological algebras has a natural

topology & we complete w.r.t. that topology.

Idea of proof:

Let $x, y \in V$ be s.t. $\omega(x,y) = 1$. We inductively construct lifts $x_k, y_k \in \hat{\mathfrak{m}}_\hbar$ s.t. $\{x_k, y_k\} - 1 \in \hat{\mathfrak{m}}_\hbar^k$ & $x_{k+1} - x_k, y_{k+1} - y_k \in \hat{\mathfrak{m}}_\hbar^{k+1}$. Take arbitrary x_1, y_1 in preimages of x, y. For the inductive step we use:

Observation: $\forall$ section $i : \hat{\mathcal{A}}_\hbar / \hat{\mathfrak{m}}_\hbar^k \longrightarrow \hat{\mathcal{A}}_\hbar / \hat{\mathfrak{m}}_\hbar^{k+1}$ of the natural projection, we have $\{x_k, y_k \cdot \operatorname{im} i\} = \hat{\mathcal{A}}_\hbar / \hat{\mathfrak{m}}_\hbar^k$. Same if we swap x_k, y_k.

Using this observation we can do the following:

- Construct x_k, y_k and hence their limits $\hat{x}, \hat{y} \in \hat{\mathfrak{m}}_\hbar$ w. $\{\hat{x}, \hat{y}\} = 1$.
- Let $\mathcal{A}_\hbar^1$ denote the centralizer of $\hat{x}, \hat{y}$ in $\hat{\mathcal{A}}_\hbar$. Then an analog of (3) works for the completed Weyl algebra generated by $\hat{x}, \hat{y}$ & the centralizer of $\hat{x}, \hat{y}$ in $\hat{\mathcal{A}}_\hbar$.

Once this is done we can argue by induction on $\dim V$. $\square$

Remarks: 1) Similar ideas lead to the following: for any 2 embeddings ι_1, ι_2 as in the theorem $\exists\, f \in \hat{\mathcal{A}}_\hbar$

- $\mathrm{ad}(f)^k a \xrightarrow[k\to\infty]{} 0 \ \forall\, a \in \hat{\mathcal{A}}_\hbar$ (where $\mathrm{ad}(f) = \{f, \cdot\}$)
- $\exp(\{f, \cdot\})\,\iota_1 = \iota_2$.

In this way $\mathcal{A}'_\hbar$, the quantum slice, is well-defined (i.e. independent from the choice of ι)

2) Assume $\mathcal{A}_\hbar$ is graded & $\mathfrak{m} \subset A$ is homogeneous, $\deg \hbar = 1$. Then ι can be chosen to be $\mathbb{C}^\times$-equivariant. In particular, $\mathcal{A}'_\hbar$ acquires a $\mathbb{C}^\times$-action.

3) Finite W-algebras

We want to apply the theorem in the following situation. Let $\mathfrak{g}$ be a s/simple Lie algebra. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form. Pick a nilpotent element $e \in \mathfrak{g}$ (recall that being nilpotent means that it is represented by a nilpotent operator in any equivalently some faithful representation). We take $A = S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ & $\mathfrak{m}$ to be the maximal ideal of $e \in \mathfrak{g} (\simeq \mathfrak{g}^*)$.

The filtered quantization we consider is $\mathcal{A} := \mathcal{U}(\mathfrak{g})$ but with the doubled filtration: $\mathcal{A}_{\leq i} := \mathcal{U}(\mathfrak{g})_{\leq [i/2]}$ so that $\mathfrak{g}$ is in deg 2 (reasons for this will be mentioned later)

So $R_\hbar(\mathcal{A}) = T(\mathfrak{g})[\hbar]/(x \otimes y - y \otimes x - \hbar^2 [x,y])$ w. $\deg \hbar = 1$ & $\mathcal{A}_\hbar$ is the $\hbar$-adic completion of this algebra.

Let's describe $\tilde{V}, \omega$ & V: $\tilde{V} = \mathfrak{g}$ w $\omega(\xi, \eta) = (e, [\xi, \eta])$ so that $\operatorname{rad} \omega = \ker(\operatorname{ad} e)$. Now a basic result in the study of nilpotent elements, the Jacobson-Morozov theorem states that $\exists\ h, f \in \mathfrak{g}$ s.t. $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$. We can take $V = [\mathfrak{g}, f]$, the representation theory of $\mathfrak{sl}_2$ tells us that this is a complement to $\ker(\operatorname{ad} e)$. And we can introduce a grading on A making $\mathfrak{m}$ homogeneous. Let $\mathfrak{g}(i) = \{\xi \in \mathfrak{g} \mid [h, \xi] = i\xi\}$ for $i \in \mathbb{Z}$. We grade $S(\mathfrak{g})$ by requiring $\deg \xi = i+2\ \forall\ \xi \in \mathfrak{g}(i)$

Since for $\xi \in \mathfrak{g}(i)$ we have $(\xi, e) \neq 0 \Rightarrow \xi \in \mathfrak{g}(-2)$, $\mathfrak{m}$ is indeed homogeneous. For this grading however $\deg\{\cdot,\cdot\}=-2$, which is why we work with the modified version of quantization above.

We can then form the algebra $\hat{\mathcal{A}}_\hbar$ w. $\mathbb{C}^\times$-action and apply the graded version of the theorem getting an algebra $\mathcal{A}'_\hbar$ also with action of $\mathbb{C}^\times$. Note that $\hbar$ has degree 1 & for the maximal ideal $\mathfrak{m}'_\hbar \subset \mathcal{A}'_\hbar$ we have (by part (3) of Thm) that $\mathfrak{m}'_\hbar/\mathfrak{m}'^2_\hbar \underset{\mathbb{C}^\times}{\simeq} \tilde{V}/V \oplus \mathbb{C}\hbar = \mathfrak{g}/[\mathfrak{g},f] \oplus \mathbb{C}\hbar$. This space is positively graded

Important exercise: Let $\mathcal{W}_\hbar \subset \mathcal{A}'_\hbar$ be the graded subalgebra $\mathcal{W}_\hbar = \bigoplus_{i\in\mathbb{Z}} \mathcal{W}_{\hbar,i}$ w. $\mathcal{W}_{\hbar,i} = \varprojlim_k (\mathcal{A}'_\hbar/\mathfrak{m}'^k_\hbar)_i$. Show that

- $\mathcal{W}_\hbar/\hbar\mathcal{W}_\hbar \xrightarrow{\sim} S(\tilde{V}/V)$ as graded algebra
- $\mathcal{A}'_\hbar$ is the completion of $\mathcal{W}_\hbar$ at 0.

The filtered algebra $\mathcal{W} := \mathcal{W}_\hbar/(\hbar-1)$ is called the finite W-algebra. It should be thought of as the filtered quantization of $\mathbb{C}[S]$, where $S = e + \ker(\operatorname{ad} f)$ is the Slodowy slice, a transverse slice to Ge (for suitable grading & Poisson structure on $\mathbb{C}[S]$, the grading is similar to what was explained above).