Lecture 4

1) Quantizations of schemes
2) Classification in symplectic setting

1) Quantizations of schemes

Let X be a scheme over $\mathbb{C}$. We can talk about Poisson structure on X = Poisson bracket on $\mathcal{O}_X$. We then can talk about formal quantizations of X (i.e. of $\mathcal{O}_X$): these are sheaves of $\mathbb{C}[[\hbar]]$-algebras $\mathcal{D}_\hbar$ on X s.t.

- $\hbar$ is not a zero divisor in $\mathcal{D}_\hbar(U)$ $\forall$ open $U \subset X$
- $\mathcal{D}_\hbar \xrightarrow{\sim} \varprojlim_n \mathcal{D}_\hbar/(\hbar^n)$
- We have a fixed isomorphism of sheaves of algebras $\mathcal{D}_\hbar/\hbar\mathcal{D}_\hbar \xrightarrow{\sim} \mathcal{O}_X$ that is Poisson (w.r.t. $\{\cdot,\cdot\}$ on the l.h.s. induced by $\frac{1}{\hbar}[\cdot,\cdot]$).

One can show that, for affine X, formal quantizations of X are in bijection w. formal quantizations of $\mathbb{C}[X]$: in one direction we take the global sections, in the other direction we (micro)localize using results of Lec 2: if $\mathcal{A}_\hbar$ is a formal quantization of $\mathbb{C}[X]$, then $\exists!$ sheaf of $\mathbb{C}[[\hbar]]$-algebras $\mathcal{D}_\hbar$ on X w. $\mathcal{D}_\hbar(X_f) = \mathcal{A}_\hbar[f^{-1}]$ $\forall$ $f \in \mathbb{C}[X]$.

Using the bijection in the affine case, we can view a formal quantization of X as glued from algebras of sections on an affine open cover.

Now suppose that X is equipped w. a $\mathbb{C}_m$-action s.t. $t.\{\cdot,\cdot\} = t^{-d}\{\cdot,\cdot\}$ $\forall$ $t \in \mathbb{C}_m$ $(d \in \mathbb{Z}_{>0})$. One can talk about graded formal quantizations of X using the usual formalism of equivariant sheaves. Alternatively if X can be covered by $\mathbb{C}_m$-stable affine opens (happens to be the case when X is quasi-projective & normal) we can glue a graded formal quantization of X from graded formal quantizations of the algebras of functions on the open pieces using $\mathbb{C}_m$-equivariant gluing maps.

Using the correspondence between filtered & graded formal quantizations, we can then talk about filtered quantizations of Poisson schemes w. compatible $\mathbb{C}_m$-action

Example: Let X_0 be a smooth variety & $X = T^*X_0$, let $\pi: X \to X_0$ be the projection. Then $\pi_* \mathcal{O}_X$ is a sheaf of graded Poisson algebras on X_0 and we can talk about filtered quantizations of this sheaf, these are so called sheaves of TDO (twisted differential operators), $\mathcal{D}_{X_0}$ is one of them. For $\mathcal{D}$, a sheaf of TDO, we can consider $\mathcal{D}_\hbar$, the $\hbar$-adic completion of $R_\hbar(\mathcal{D})$, still a sheaf on X_0. But we can microlocalize it to obtain a sheaf on X: this is a special case of the previous construction, where we consider cover $X = \bigcup_i T^*X_0^i$ for open affine cover $X = \bigcup_i X_0^i$. We can also consider the corresponding sheaf of filtered algebras but the sections are only going to be defined on $\mathbb{C}^\times$-stable open subsets.

2) Classification in symplectic setting

Till the end of the lecture assume X is smooth & symplectic. We are interested in classifying (graded, if applicable) formal quantizations of X.

General principle: Under suitable cohomology vanishing conditions on $\mathcal{O}_X$, the formal quantizations are classified by $H^2_{DR}(X)[[\hbar]]$ $(= H^2(X,\mathbb{C})[[\hbar]])$ & graded formal quantizations are classified by $H^2_{DR}(X)$.

2.1) Warm-up: classification of sheaves of TDO.

We state this as a long exercise:

Exercise: 1) Let X_0 be a smooth affine variety & $\omega \in \Omega^2(X)$ be a closed form. Define the algebra $D_\omega(X_0)$ by generators $\mathbb{C}[X_0]$, $\mathrm{Vect}(X_0)$ & the following relations:

- $f * g = fg$
- $f * \xi = f\xi$
- $\xi * f = f\xi + \xi.f$
- $\xi * \eta - \eta * \xi = [\xi,\eta] + \omega(\xi,\eta)$

$\forall\, f,g \in \mathbb{C}[X_0]$, $\xi,\eta \in \mathrm{Vect}(X_0)$

Show that this is an algebra of TDO (which reduces to the claim that $\mathbb{C}[T^*X_0] \twoheadrightarrow \mathrm{gr}\, D_\omega(X_0)$ is iso; hint:

base change to formal nghds of pts in X_0 + part 3 below)

2) Prove that every algebra of TDO on X_0 is isomorphic to $\mathcal{D}_\omega(X_0)$ for some closed $\omega \in \Omega^2(X_0)$

3) Let ω_1, ω_2 be closed. For any isomorphism $\varphi: \mathcal{D}_{\omega_1}(X_0) \to \mathcal{D}_{\omega_2}(X_0)$ of filtered quantizations (i.e. filtered algebra isomorphism s.t. $\operatorname{gr}\varphi$ intertwines the identifications $\operatorname{gr}\mathcal{D}_{\omega_i}(X_0) \xrightarrow{\sim} \mathbb{C}[T^*X_0]$) $\exists!\ \alpha \in \Omega^1(X_0)$ s.t. $\varphi(f) = f$, $\varphi(\xi) = \xi + \langle \alpha, \xi \rangle$ $\forall f \in \mathbb{C}[X_0]$, $\xi \in \operatorname{Vect}(X_0)$. This α satisfies $d\alpha = \omega_2 - \omega_1$. Conversely, such α gives an isomorphism.

In particular, algebras of TDO are indeed classified by $H^2_{DR}(X_0) = H^2_{DR}(T^*X_0)$.

4) Now let X_0 be an arbitrary smooth variety. Let $\Omega^{\geqslant 1}_{X_0}$ be the truncated de Rham complex $\underset{\text{deg } 0}{0} \to \Omega^1_{X_0} \to \Omega^2_{X_0} \to \ldots$ Then sheaves of TDO are classified by $H^2(\Omega^{\geqslant 1}_{X_0})$ (= 2nd cohomology of the Čech-de Rham complex). Hint: glue from open affines.

5) Every filtered quantization of T^*X_0 arises from a unique sheaf of TDO (via microlocalization).

Rem: Note that we have a SES of complexes

$0 \to \Omega^{\geqslant 1}_{X_0} \to \Omega_{X_0} \to \mathcal{O}_{X_0} \to 0$ giving exact sequence

$$H^1(\mathcal{O}_{X_0}) \to H^2(\Omega^{\geqslant 1}_{X_0}) \to H^2_{DR}(X_0) \to H^2(\mathcal{O}_{X_0})$$

In particular if $H^1(\mathcal{O}_{X_0}) = H^2(\mathcal{O}_{X_0})$, then $H^2(\Omega^{\geqslant 1}_{X_0}) = H^2_{DR}(X_0)$

Note that $\mathcal{O}_{X_0}$ is the direct summand (deg 0 component) in the graded sheaf $\pi_* \mathcal{O}_{X_0}$, hence $H^i(\mathcal{O}_{X_0}) \subset H^i(\pi_* \mathcal{O}_X) = H^i(\mathcal{O}_X)$

In particular, if $H^i(\mathcal{O}_X) = 0$ for $i = 1, 2$, then sheaves of TDO are classified by $H^2_{DR}(X)$ confirming the general principle in the beginning of the lecture.

Example: $X_0 = G/B$, where G is a s/simple algebraic group & $B \subset G$ is a Borel subgroup. Let $\mathfrak{h} := \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ be the universal Cartan. Then $H^2_{DR}(X_0) = \mathfrak{h}^*$. One can construct the sheaf of TDO corresponding to λ as follows. Let $U := Rad_u(B)$ so that $Lie(U) = [\mathfrak{b},\mathfrak{b}]$. Let $\varpi: G/U \to G/B$ be the projection. Note that $G \curvearrowright G/U \curvearrowleft H$. The sheaf $\varpi_* \mathcal{D}_{G/U}$ has H-action by algebra automorphisms & we set $\mathcal{D}^{univ}_{G/B} := (\varpi_* \mathcal{D}_{G/U})^H$. This is a sheaf of $S(\mathfrak{h})$-algebras: the action of H on G/U gives rise to a Lie algebra homomorphism $\mathfrak{h} \to \mathrm{Vect}(G/U)$ w. H-invariant image, so it extends to $S(\mathfrak{h}) \to \Gamma((\varpi_* \mathcal{D}_{G/U})^H)$. The sheaf of TDO corresponding to λ is

$$\mathcal{D}^{\lambda}_{G/B} := \mathcal{D}^{univ}_{G/B} \otimes_{S(\mathfrak{h})} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the quotient of $S(\mathfrak{h})$ by the maximal ideal of λ.

2.2) Classification of quantizations

Thm Let X be a smooth symplectic variety w. $H^i(X, \mathcal{O}_X)=0$ for $i=1,2$. Then the following hold:

1: (Bezrukavnikov & Kaledin) The formal quantizations of X are parameterized by $H^2_{DR}(X)[[\hbar]]$ in a natural way

2: (I.L.) Suppose in addition that $\mathbb{C}^\times$ acts on X s.t. $\{\cdot,\cdot\}$ has degree $-d$ (for $d>0$). Then the graded formal quantizations of X are naturally parameterized by $H^2_{DR}(X)$.

Example: The graded formal quantization of $T^*(G/B)$ corresponding to λ is $\mathcal{D}^{\lambda-\rho}_{G/B}$, where ρ is half the sum of positive roots.

Something about a proof: here are a few related observations. First there's only one quantization of the completion $\widehat{\mathbb{C}[X]_x}$ $\forall x \in X$, the formal Weyl algebra $\hat{W}_\hbar(T_x X)$, see the quantum slice theorem in Sec 2 of Lec 3. One can describe its automorphisms as quantization of $\hat{\mathcal{O}}_{X,x}$ completion they are of the form $\exp(\hbar\partial)$ for $\partial \in \mathrm{Der}(\hat{W}_\hbar(T_x X))$ (where ∂ is uniquely recovered from $\exp(\hbar\partial)$). Moreover, we have a short exact sequence

$$0 \to \mathbb{C}[[\hbar]] \longrightarrow \hat{W}_\hbar(T_x X) \xrightarrow{a \mapsto \{a,\cdot\}} \mathrm{Der}(\widehat{\mathbb{C}[X]_x}) \longrightarrow 0$$

It's the first term whose 2nd cohomology parameterizes the quantizations: there's a framework for gluing the quantizations of $\hat{\mathcal{O}}_{X,x}$'s together, since $H^i(X, \mathcal{O}_X) = 0$ $\forall i = 1, 2$, this gluing is unobstructed and is controlled by the 2nd cohomology of $\mathbb{C}[[\hbar]]$.

For the 2nd part, one thing that is easy to see is that the parameter of a graded formal quantization is in $H^2_{DR}(X) \subset H^2_{DR}(X)[[\hbar]]$. Roughly this is because the parameterization is natural, while the natural action of $\mathbb{C}^\times$ on $H^2_{DR}(X)$ is trivial, while $\deg \hbar = 1$. And of course, any graded formal quantization is a fixed point for the action of $\mathbb{C}^\times$ on the set of isomorphism classes of quantizations. $\square$