Lecture 5

1) Conical symplectic singularities
2) Classification of quantizations

1) Conical symplectic singularities

In Lec 4 we have stated the classification of (graded) formal quantizations of smooth symplectic varieties. This is not what is interesting in Rep. theory, where we are dealing w. graded algebras that are algebras of functions on singular affine varieties. An example of such is the nilpotent cone $\mathcal{N}$ in $\mathfrak{g}^*(\simeq\mathfrak{g})$, where $\mathfrak{g}$ is a semisimple Lie algebra (by definition, $\mathcal{N}$ is the subvariety of all nilpotent elements). It turns out, however, that for many singular varieties the classification reduces to that for suitable smooth varieties (for $\mathcal{N}$ the corresponding smooth symplectic variety is $T^*(G/B)$). The correct generality here is when the singular variety in question is a conical symplectic singularity.

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1.1) Definition

Let X be a Poisson variety.

Definition (Beauville)

We say that X is singular symplectic if

(i) X is normal

(ii) X^{reg} is symplectic (so that the restriction of the Poisson structure to X^{reg} is induced by the symplectic form). Denote this form by ω^{reg}.

(iii) $\exists$ resolution of singularities $\tilde{Y} \xrightarrow{\rho} X$ s.t. $\rho^*(\omega^{reg})$ extends to a regular (but possibly degenerate) form on $\tilde{Y}$. Note that once this condition holds for one resolution, it holds for any.

Definition: By a conical symplectic singularity we mean an affine singular symplectic variety X together w. a $\mathbb{Z}_{\geq 0}$-grading on $\mathbb{C}[X]$ s.t. $\mathbb{C}[X]_0 = \mathbb{C}$ & $\deg\{\cdot,\cdot\} = -d$ (for $d \in \mathbb{Z}_{>0}$).

1.2) Examples

1) $X = \mathcal{N} \subset \mathfrak{g}^*$, Poisson subvariety equipped w. dilation action

of $\mathbb{C}^\times$ (so that $d=1$), $\tilde{Y}=T^*(G/B)$ & ρ is Springer resolution (a.k.a. the moment map for $G\curvearrowright\tilde{Y}$); $\mathcal{N}$ is known to be normal. The smooth locus is the open orbit in $\mathcal{N}$ called principal. This locus is symplectic. One can show that ρ is a Poisson morphism so $\rho^*\omega^{reg}$ extends to the tautological symplectic form on $\tilde{Y}$.

2) Let $\mathbb{O}\subset\mathcal{N}$ be a G-orbit. Recall, Sec 3 in Lec 3, that we have the transverse (Slodowy) slice $S\subset\mathfrak{g}^*$ to $\mathbb{O}$. Take $X:=S\cap\mathcal{N}$, $\tilde{Y}:=T^*(G/B)\times_{\mathfrak{g}^*}S$. From the transversality condition X is normal. In Sec 3 of Lec 3 we have introduced a $\mathbb{C}^\times$-action on S s.t. $\mathbb{C}[S]$ is $\mathbb{Z}_{\geq 0}$-graded w. $\mathbb{C}[S]_0=\mathbb{C}$. It's easy to check that $\mathbb{C}[X]$ is a graded quotient. Next we get $\{\cdot,\cdot\}$ on $\mathbb{C}[S]$ say from its quantization $\mathcal{W}$ constructed in Sec 3 of Lec 3 (one can also do an easier version in the Poisson setting); $\deg\{\cdot,\cdot\}=-2$.

Exercise:

1) $S(\mathfrak{g})^G\subset S(\mathfrak{g})$ is Poisson central

2*) The restrictions of elements of $S(\mathfrak{g})^G$ to S lie in the Poisson center of $\mathbb{C}[S]$.

This gives rise to a Poisson bracket of deg -2 on $\mathbb{C}[X]$ (as a Poisson quotient of $\mathbb{C}[S]$).

Required properties of $\tilde{Y}$ (smooth, symplectic - in fact as a subvariety of $T^*(G/B)$, and $\rho: \tilde{Y} \to X$ is Poisson) all can be deduced from the fact that ρ is a moment map & S is transverse to all orbits that it intersects.

So $X = S \cap \mathcal{N}$ is a conical symplectic singularity.

3) In the notation of Example 2) consider the closure $\overline{\mathbb{O}}$, a Poisson subvariety of $\mathfrak{g}^*$. It's not normal, so let X denote the normalization. One can show that $\mathbb{C}[\mathbb{O}] = \mathbb{C}[X]$ (by using the observation that the number of orbits in $\overline{\mathbb{O}}$ is finite & all have even dimension). The dilation action of $\mathbb{C}^\times$ on $\mathfrak{g}^*$ preserves $\overline{\mathbb{O}}$, hence lifts to $\mathbb{C}[X]$. The symplectic form on $\mathbb{O}$ gives rise to a deg -1 Poisson bracket on $\mathbb{C}[X]$. Note that $\operatorname{codim}_{X^{reg}} X^{reg} \setminus \mathbb{O} \geq 2$, & $\mathbb{O}$ is symplectic so is X^{reg}. It is known after Panyushev

that X is singular symplectic, hence a conical symplectic singularity.

1.3) $\mathbb{Q}$-factorial terminalization.

In general, a singular symplectic variety does not admit a symplectic resolution (i.e $(\tilde{Y},\rho)$ s.t. $\tilde{Y}$ is symplectic & ρ is Poisson). In Examples 1&2 we have such resolutions, while in Example 3 we do not, in general. However, in all cases X admits a normal Poisson partial resolution (Y,ρ) which is maximal (w.r.t. the natural order on partial resolutions) equivalently, Y is "$\mathbb{Q}$-factorial" (every Weil divisor admits a multiple which is Cartier) & "terminal." The latter in our setting means:

(1) $\operatorname{codim}_Y Y^{sing} \geqslant 4$

Here are important properties of Y.

Proposition: 1) $H^0(\mathcal{O}_{Y^{reg}}) = \mathbb{C}[X]$ & $H^i(\mathcal{O}_{Y^{reg}})=0$ $\forall\, i=1,2$.

2) The $\mathbb{C}^\times$-action on X (uniquely) lifts to Y.

Sketch of proof: 2) is due to Namikawa. Let's explain why 1) holds. A result of Beauville says that symplectic singularities are rational: if X' is singular symplectic & $\rho': Y' \to X'$ is a proper birational morphism from a smooth variety, then $R\rho'_* \mathcal{O}_{Y'} = \mathcal{O}_{X'}$. Now take a resolution of singularities $\tilde{Y} \xrightarrow{\rho'} Y$ so that $\rho\circ\rho': \tilde{Y} \to X$ is a resolution of singularities as well. By Def'n in Sec 1.1, the symplectic form on X^{reg} lifts to $\tilde{Y}$ from which it's not hard to conclude that Y is singular symplectic. So $R\rho'_* \mathcal{O}_{\tilde{Y}} = \mathcal{O}_Y$ & $R(\rho\circ\rho')_* \mathcal{O}_{\tilde{Y}} = \mathcal{O}_X \Rightarrow R\rho_* \mathcal{O}_Y = \mathcal{O}_X$.

Let $\iota: Y^{reg} \hookrightarrow Y$ be the inclusion. Rational singularities are Cohen-Macaulay $\Rightarrow R^0\iota_* \mathcal{O}_{Y^{reg}} = Y$ & $R^i\iota_* \mathcal{O}_{Y^{reg}} = 0$ for $1 \le i \le \operatorname{codim}_Y Y^{sing} - 2$. Since $\operatorname{codim}_Y Y^{sing} \geqslant 4$ we are done. $\square$

In particular, Theorem from Lec 4 can be applied to classify graded formal quantizations of Y^{reg}: they are classified by $H^2(Y^{reg}, \mathbb{C}) = H^2_{DR}(Y^{reg})$

2) Classification of quantizations

Let X be a conical symplectic singularity & Y be its $\mathbb{Q}$-factorial terminalization. By the Cartan space of X we mean $\mathfrak{h}_X := H^2(Y^{reg}, \mathbb{C})$. Note that, in general, Y is not unique but these spaces for different Y can be identified.

Thm: $\exists$ crystalographic reflection group $W_X \subset GL(\mathfrak{h}_X)$ s.t. we have a natural bijection:

$$\{\text{filtered quantizations}\}/\text{iso} \xrightarrow{\sim} \mathfrak{h}_X/W_X \left(= \operatorname{Spec} \mathbb{C}[\mathfrak{h}_X]^{W_X}\right)$$

Example: Let $X = \mathcal{N} (\subset \mathfrak{g}^*)$. The quantizations are constructed as quotients of $\mathcal{U}(\mathfrak{g})$ as follows. The HC isomorphism states that the center of $\mathcal{U}(\mathfrak{g})$ is identified with $\mathbb{C}[\mathfrak{h}^*]^W$. So for $\lambda \in \mathfrak{h}^*/W$ we have the maximal ideal $\mathfrak{m}_\lambda \subset \mathcal{U}(\mathfrak{g})$. Set $\mathcal{A}_\lambda := \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{m}_\lambda$. It inherits the filtration from $\mathcal{U}(\mathfrak{g})$ and one can show that it becomes a filtered quantization of $\mathbb{C}[\mathcal{N}]$. Here $\mathfrak{h}_X := \mathfrak{h}^*$, $W_X = W$.

Recall that filtered quantizations of $Y = T^*(G/B)$ are classified by $H^2(Y,\mathbb{C})=\mathfrak{h}^*$: to $\tilde{\lambda}\in\mathfrak{h}^*$ we assign the sheaf $\mathcal{D}^{\lambda-\rho}_{G/B}$ of $(\lambda-\rho)$-twisted differential operators on G/B. It's well-known that $\mathcal{A}_\lambda := \Gamma(\mathcal{D}^{\tilde{\lambda}-\rho}_{G/B})$.

The construction of $\mathcal{A}_\lambda$ w. $\lambda\in\mathfrak{h}_X/W_X$ in the general case is similar: for $\tilde{\lambda}\in\mathfrak{h}_X$ we take the graded formal quantization $\mathcal{D}_{\tilde{\lambda}}$ of Y^{reg}. Thx to $\Gamma(\mathcal{O}_{Y^{reg}})=\mathbb{C}[X]$ & $H^1(\mathcal{O}_{Y^{reg}})=0$, $\Gamma(\mathcal{D}_{\tilde{\lambda}})$ is a graded formal quantization of $\mathbb{C}[X]$ (exercise). Turns out that these quantizations can be isomorphic for different $\tilde{\lambda}$, this is controlled by $W_{\tilde{\lambda}}$. Then for $\lambda:=W_X\tilde{\lambda}$ we take the filtered quantization $\mathcal{A}_\lambda$ corresponding to $\Gamma(\mathcal{D}_{\tilde{\lambda}})$.