1 Equivariant integrals over noncompact spaces

Let Y be a proper variety with an action of the group G. Let $\gamma \in H^*_G(Y)$. The equivariant integral

$$\int_Y \gamma \in H^*_G(pt)$$

is simply the equivariant pushforward of γ to a point. It is linear over $H^*_G(pt)$. Using the Atiyah-Bott localization formula from Ryan Mickler's talk, we can extend the definition of the equivariant integral to certain noncompact spaces:

Definition: Let Y be smooth variety with an action of a torus T, such that Y^T is proper. Let $\gamma \in H^*_T(Y)$. Let F_i be the fixed loci of T, with $\iota_{F_i} : F_i \to Y$ the inclusion maps. We define the equivariant integral

$$\int_Y \gamma = \sum_i \int_{F_i} \frac{\iota^*_{F_i}\gamma}{Eu(N_{F_i}Y)} \in H^*_G(pt)_{loc}. \tag{1}$$

Note that the integral now lands in $H^*_G(pt)_{loc}$, i.e. the ring of fractions of $H^*_G(pt)$. The Atiyah-Bott formula shows when Y is proper, this definition matches the usual integral and lands in $H^*_G(pt)$.

Example:. Consider $\mathbb{C}$ with the natural action of $\mathbb{C}^*$. Then

$$\int_{\mathbb{C}} 1 = \frac{\iota_0^* 1}{Eu(T_0\mathbb{C})} = \frac{1}{a}$$

where a is a generator of $H^*_{\mathbb{C}^*}(pt)$. The noncompactness of $\mathbb{C}$ is manifested by the appearance of nontrivial denominators in the integral.

We note a useful property of the equivariant integral for further use:

Proposition: Let X be an n-dimensional variety with G-action, and let $\gamma \in H^k_G(X)$. If X is proper and $k < n$, then $\int_X \gamma = 0$.

Note that if X is not proper, this may no longer hold. Note also that the opposite inequality does not ensure vanishing of the integral, contrary to the nonequivariant case.

2 Equivariant Gromov-Witten invariants

Let X be a smooth proper variety with a action of a group G. Recall that the Gromov-Witten invariants of X are rational numbers

$$\langle \gamma_1, \ldots, \gamma_n \rangle^X_{g,n,\beta} \in \mathbb{Q}$$

They are defined by the integral

$$\int_{\bar{\mathcal{M}}_{g,n}(X,\beta)} ev_1^*\gamma_1 \cup \ldots ev_n^*\gamma_n. \tag{2}$$

One can define an equivariant virtual fundamental class [?], living in $H_*^G(\bar{\mathcal{M}}_{g,n}(X,\beta))$. If we let $\gamma_1,\ldots,\gamma_n$ live in $H_G^*(X)$, then we can define the equivariant Gromov-Witten invariant by the same formula, where the integral becomes equivariant:

$$\langle\gamma_1,\ldots,\gamma_n\rangle_{g,n,\beta}^{X,G} \in H_G^*(pt,\mathbb{Q}).$$

In fact, this allows to extend the range of Gromov-Witten theory to non-compact varieties with compact fixed locus under the action of a torus T.

Definition: Let X be a possibly non-compact smooth variety with G-action, such that $T \subset G$ has proper fixed locus. Let $\gamma_i \in H_G^*(X)$, $\beta \in H_2(X,\mathbb{Z})$ The equivariant Gromov-Witten invariant

$$\langle\gamma_1,\ldots,\gamma_n\rangle_{g,n,\beta}^{X,G} \in H_G^*(pt,\mathbb{Q})_{loc}$$

is defined by using 1 to define the integral in 2.

This extension will not be used in today's talk, but will be handy in future talks.

3 Equivariant Small Quantum Cohomology

Equipped with equivariant Gromov-Witten invariants, it is now straightforward to define the equivariant quantum cohomology of X, $QH_G^*(X)$, as a commutative associative deformation of $H_G^*(X,\mathbb{C})$. The quantum product is determined by

$$(\gamma_1 * \gamma_2,\gamma_3)^G = \sum_\beta \langle\gamma_1,\gamma_2,\gamma_3\rangle_{0,3,\beta}^{X,G} q^\beta.$$

$QH_G^*(X)$ is a module over $H_G^*(pt)$; in other words, multiplication by purely equivariant classes is not subject to quantum corrections. Finally, note that the deformation parameter still lives in $H^2(X,\mathbb{C})/H^2(X,\mathbb{Z})$, not $H_G^2(X,\mathbb{C})/H_G^2(X,\mathbb{Z})$.

4 Equivariant Quantum Cohomology of $\mathbb{P}^1$

Consider $\mathbb{P}^1$ with $\mathbb{C}^*$ acting by $[x:y] \to [z^{-1}x:y]$. Let $u = c_1(\mathcal{O}(1))$, where $\mathcal{O}(1)$ is given the natural linearization. Set $H_{\mathbb{C}^*}^*(pt) = \mathbb{C}[a]$. Then it follows from the Leray-Hirsch formula for the cohomology of a $\mathbb{P}^1$ bundle that

$$H_{\mathbb{C}^*}^*(\mathbb{P}^1) = \mathbb{C}[u,a]/u(u-a).$$

The equivariant integral is determined by

$$\int_{\mathbb{P}^1} 1 = 0, \tag{3}$$

$$\int_{\mathbb{P}^1} u = 1. \tag{4}$$

Effective curve classes in $\mathbb{P}^1$ are given by multiples nl of the line l. Hence to determine $QH_{\mathbb{C}^*}^*(\mathbb{P}^1)$, we need only determine

$$\langle u,u,1\rangle_{0,3,nl}^{X,G} \quad n = 0,1,2,\ldots \tag{5}$$

$$\langle u,u,u\rangle_{0,3,nl}^{X,G} \quad n = 0,1,2,\ldots \tag{6}$$

All other invariants are determined by linearity over $H^*_{\mathbb{C}^*}(pt)$ and by the fact that 1 remains a unit in quantum cohomology. We have

$$dim_{\mathbb{C}}[\mathcal{M}_{0,3}(\mathbb{P}^1, nl)]^{vir} = 1 + 2n.$$

Comparing the degree of the integrand to the dimension of the space, we conclude that all invariants vanish except possibly

$$\langle u, u, 1\rangle^{X,G}_{0,3,0} \tag{7}$$

$$\langle u, u, u\rangle^{X,G}_{0,3,l} \tag{8}$$

We have

$$\langle u * u, 1\rangle^{X,G}_{0,3,0} = \int_{\mathbb{P}^1} u^2 = \int au = a.$$

To compute 15, we apply the divisor equation three times to get

$$\langle u, u, u\rangle^{X,G}_{0,3,l} = \langle\rangle^{X,G}_{0,0,l} = 1$$

as there is a unique unpointed rational map to $\mathbb{P}^1$, up to automorphism. We conclude

$$u * u = au + q$$

and

$$QH^*_{\mathbb{C}^*}(\mathbb{P}^1) = \mathbb{C}[u, a]/u(u - a) - q$$

5 Quantum cohomology of $T^*\mathbb{P}^1$

Consider the action of $T = \mathbb{C}^* \times \mathbb{C}^*$ on $T^*\mathbb{P}^1$ where the first factor acts by $[x : y] \to [z^{-1}x : y]$ on $\mathbb{P}^1$ and by the induced action on $T^*\mathbb{P}^1$, and the second factors simply dilates the cotangent fibers. Write $H^*_T(pt) = \mathbb{C}[a, \hbar]$, where a corresponds to the first factor, $\hbar$ to the second. $T^*\mathbb{P}^1$ retracts equivariantly onto $\mathbb{P}^1$, whence it follows that

$$H^*_T(T^*\mathbb{P}^1) = \mathbb{C}[\hbar, a, u]/(u(u - a)).$$

T has two fixed points, at $[1, 0]$ and $[0, 1]$. The reader may verify that the normal bundles have equivariant euler classes

$$Eu(N_{[1,0]}X) = a(\hbar - a) \tag{9}$$

$$Eu(N_{[0,1]}X) = -a(\hbar + a) \tag{10}$$

The reader may check that $u|_{[1,0]} = a, u|_{[0,1]} = 0$. The integral is thus given by

$$\int_X 1 = \frac{1}{a(\hbar - a)} + \frac{1}{-a(\hbar + a)} \tag{11}$$

$$\int_X u = \frac{a}{a(\hbar - a)} \tag{12}$$

$$\tag{13}$$

The space of rational maps to $T^*\mathbb{P}^1$ is the same as that for $\mathbb{P}^1$: all maps land in the zero section. The Gromov-Witten theory is different, however, since the virtual fundamental class is different: it is no longer the fundamental class of $\bar{\mathcal{M}}_{0,n}(X, nl)$.

5.1 Non-equivariant quantum cohomology

We have

$$dim_{\mathbb{C}}[\mathcal{M}_{0,3}(T^*\mathbb{P}^1, nl)]^{vir} = dim_{\mathbb{C}}(T^*\mathbb{P}^1) + c_1(T^*\mathbb{P}^1)(nl) + 3 - 3 = 2.$$

This means that $\langle \gamma_1, \gamma_2, \gamma_3 \rangle^X_{0,3,ml}$ vanishes unless one of the insertions $\gamma_i = 1$. This forces $n = 0$, hence the ordinary quantum cohomology has no quantum corrections. Note that the same would be true of any surface S with $c_1(S) = 0$; in fact, any embedded copy of $\mathbb{P}^1$ in such a surface will have normal bundle isomorphic to $T^*\mathbb{P}^1$.

5.2 Equivariant quantum cohomology

As with $\mathbb{P}^1$, we need only compute

$$\langle u, u, 1 \rangle^{X,G}_{0,3,0} \tag{14}$$

$$\langle u, u, u \rangle^{X,G}_{0,3,nl} \tag{15}$$

We have

$$\langle u, u, 1 \rangle^{X,G}_{0,3,0} = \int_X u^2 = \frac{a^2}{a(\hbar - a)}$$

We now compute the degree one case of 15. We can again use the divisor equation to get

$$\langle u, u, u \rangle^{X,G}_{0,3,l} = \langle \rangle^{X,G}_{0,0,l}.$$

The moduli space $\bar{\mathcal{M}}_{0,0}(X, l)$ is, as before, a single point corresponding to the isomorphism $f : C \to \mathbb{P}^1$. We label the domain curve by C, though it is also a copy of $\mathbb{P}^1$, to avoid confusion. Its virtual fundamental class is defined using the deformation-obstruction sequence described in Barbara Bolognese's talk; in this case the result is

$$[\bar{\mathcal{M}}_{0,0}(X, l)]^{vir} = \frac{Eu(H^1(C, f^*TX) \oplus H^0(C, TC))}{Eu(H^0(C, f^*TX))}$$

Here $H^*(C, f^*TX)$ is thought of as an equivariant bundle over the point. The terms in the numerator record obstructions and automorphisms, whereas the denominator records deformations. We have

$$f^*TX = \mathcal{O}(2) \oplus \Big(\mathcal{O}(-2) \otimes \mathbb{C}_\hbar\Big) = TC \oplus \Big(T^*C \otimes \mathbb{C}_\hbar\Big)$$

where $\mathbb{C}_\hbar$ is the defining representation of the $\mathbb{C}^*$ which dilates the cotangent fibers. Hence

$$H^1(C, f^*TX) = H^1(C, \mathcal{O}(-2) \otimes \mathbb{C}_\hbar) = H^1(C, \mathcal{O}(-2)) \otimes \mathbb{C}_\hbar = \mathbb{C}_\hbar$$

and

$$[\bar{\mathcal{M}}_{0,0}(X, l)]^{vir} = \frac{Eu(\mathbb{C}_\hbar \oplus H^0(C, TC))}{Eu(H^0(C, TC))} = Eu(\mathbb{C}_\hbar) = \hbar$$

In fact, one can check that

$$\langle u, u, u \rangle^{X,G}_{0,3,nl} = \hbar$$

for any n. It follows that

$$u * u = au + \hbar \sum_n q^{nl}(\hbar + a - 2u) \tag{16}$$

and hence, after analytic continuation of the geometric series, we obtain

$$QH^*_{\mathbb{C}^* \times \mathbb{C}^*}(T^*\mathbb{P}^1) = \mathbb{C}[u, a, \hbar]/u^2 = a + \hbar \frac{q^l}{1 - q^l}(\hbar + a - 2u). \tag{17}$$

5.3 Quantum corrections via Steinberg correspondences

Let $Z = \mathbb{P}^1 \times \mathbb{P}^1 \subset T^*\mathbb{P}^1 \times T^*\mathbb{P}^1$ be the nondiagonal component of the Steinberg variety. It acts on $H^*_T(T^*\mathbb{P}^1)$ via

$$Z(\theta) = (\pi_2)_*(Z \cup \pi_1^*(\theta))$$

where π_1, π_2 are the projections to the two factors of $T^*\mathbb{P}^1 \times T^*\mathbb{P}^1$. We leave it to the reader to check, via localization, that equation 16 is equivalent to

$$u * \theta = u \cup \theta + \hbar \frac{q^l}{1 - q^l} Z(\theta).$$

Hence the quantum corrections are neatly expressed in terms of the Steinberg variety.